# COORDINATE GEOMETRY

# COORDINATE GEOMETRY

# COORDINATE GEOMETRY

C.K. Constandinides

BY

J. H. GRACE, M.A., F.R.S.

FELLOW OF ST PETER'S COLLEGE, CAMBRIDGE

AND

F. ROSENBERG, M.A. (CANTAB.), B.Sc. (LOND.)

*Fourth Impression*

LONDON: W. B. CLIVE

University Tutorial Press Ld.

HIGH ST., NEW OXFORD ST., W.C.

1919

# PREFACE.

ANALYTICAL Geometry contains confessedly so many difficulties for the ordinary beginner that any work that develops the subject in a more gradual and more explanatory manner than existing text-books has a good *raison d'être.* Noticeably is there such a lack in the treatment of curve tracing, one of the most useful branches of elementary mathematics. It is a general experience that a large proportion of students can make nothing of this portion of the subject. We believe that their failure is solely due to the defective treatment given by their text-books, and in this work we hope to remove the defect.

In the order of treatment we have followed the classical work of Salmon more closely than usual in that, after short chapters on the three varieties of conics, we proceed at once to the general equation of the second degree and the classification of curves represented by such an equation. Curve tracing is thus introduced at a comparatively early point. Afterwards, the treatment of the various properties, tangent, normal, parametric, etc., of conics is to a great extent general, *i.e.*, the methods apply to all curves of the second degree. Each of these properties is dealt with in a separate chapter. This manner of division seems to us to render the subject much easier for students, though it, like all others, has disadvantages.

As much help as is considered desirable has been given in the way of illustrative examples, numerous "cautions," and hints for the solution of the exercises. In connection with the first we would impress upon the reader the very great importance we attach to an attempt on the reader's part to solve bookwork theorems and the illustrative examples for himself previous to reading the proofs given.

In most cases the reader will not succeed, but the benefit accruing from the attempt cannot, as all experienced teachers know, be over-estimated.

The comparative importance of the sections is indicated by the size of the type in which they are printed, while the most important pieces of bookwork have also their section numbers in dark type (thus **78**). Many of the worked and unworked examples are intrinsically valuable. These are distinguished by enunciations in dark type. The formulae that should be committed to memory are indicated in the same manner.

The scope of this book is that of the London B.A. Examination, but it will also be found suitable for the following examinations:—Oxford and Cambridge Higher Certificate; Cambridge Senior and Higher Local; Oxford Senior Local; College of Preceptors Diploma; Science and Art.

# CONTENTS.

PAGES

## PART I., THE STRAIGHT LINE.

RECTANGULAR AND OBLIQUE AXES. THE STRAIGHT LINE, Equations of. Propositions relating to the lines $ax^2 + 2hxy + by^2 = 0$. Distance between two points. CHANGE OF AXES. Linear relation between old and new coordinates. Degree of equation unaltered by change of axes. Invariants ... ... ... ... 1—54

*Examination Paper* ... ... ... ... 55

## PART II., THE CIRCLE AND CONIC.

CHAPTER I.—EQUATION OF CIRCLE. Chords and Tangents. Condition of Tangency ... ... ... ... ... 1—14

CHAPTER II.—POLE AND POLAR OF CIRCLE. Length of Tangent. Radical Axis. Equation of Pair of Tangents. Inversion ... ... ... ... ... ... 15—33

*Examination Paper A*... ... ... ... ... 34

CHAPTER III.—THE PARABOLA. Mechanical description. Equation $y^2 = 4ax$. Equation referred to axes parallel to principal axes. Terms of second degree form a perfect square. Tangent to parabola ... ... ... 35—48

*Examination Paper I.* ... ... ... ... 49

CHAPTER IV.—THE ELLIPSE. Equation $x^2/a^2 + y^2/b^2 = 1$. Shape of ellipse. Equation referred to axes parallel to principal axes. Polar equation. Sum of focal distances. Mechanical description. Circle as limiting case. Tangent to ellipse. Condition for ellipse $ab > h^2$ 50—68

CHAPTER V.—THE HYPERBOLA. Equation $x^2/a^2 - y^2/b^2 = 1$. Shape of hyperbola. Equation referred to axes parallel to principal axes. Difference of focal distances. Mechanical description. Tangent to hyperbola. Equation of Asymptotes. Rectangular hyperbola. Condition for hyperbola $ab < h^2$. Equation $xy = \frac{1}{4}(a^2 + b^2)$ referred to asymptotes as axes ... 69—87

*Examination Paper II.* ... ... ... .. 88

CHAPTER VI.—GENERAL EQUATION OF THE SECOND DEGREE. Equation referred to centre. Equations and lengths of semi-axes. Asymptotes ... ... ... ... 89—105

CHAPTER VII.—TRACING OF ELLIPSES from the general equation ... ... ... .. ... ... ... 106—111

PAGES

CHAPTER VIII.—TRACING OF HYPERBOLAS from the general equation ... ... ... ... ... ... ... 112—116

CHAPTER IX.—GENERAL EQUATION when $ab = h^2$. Latus rectum and axis of parabola ... ... ... ... 117—123

CHAPTER X.—TRACING OF PARABOLAS from the general equation ... ... ... ... ... ... ... 124—126

CHAPTER XI.—Tracing of conics in general... ... ... 127—138

*Examination Paper III.* 139

CHAPTER XII.—CHORDS AND TANGENTS.—Distance quadratic. Tangents at $(x_1, y_1)$. Equation of asymptotes. Ratio quadratic. Pair of tangents. Chord and tangent. Tangents at right angles. Auxiliary circle ... 140—168

CHAPTER XIII.—MID POINTS OF PARALLEL CHORDS. CONJUGATE DIAMETERS. Conjugate hyperbola. Parabola referred to tangent and diameter as axes ... ... 169—191

CHAPTER XIV.—NORMALS. Normal to parabola in the form $y = mx - 2am - am^3$. Conic referred to tangent and normal as axes ... ... .. ... ... 192—202

*Examination Paper IV.* ... ... ... ... 203

CHAPTER XV.—POLES AND POLARS. Equations to Polars. Reciprocal relation. Geometrical construction ... 204—219

CHAPTER XVI.—PARAMETRIC REPRESENTATION. Eccentric angle. Area of ellipse. Equation of chord and tangent of ellipse. Normals ... ... ... ... 220—242

CHAPTER XVII.—POLAR EQUATION OF CONIC WITH FOCUS AS A POLE. Form of Curve. Equation of chord and tangent ... ... ... .. ... .. ... 243—253

*Examination Paper V.* ... ... ... ... 254

CHAPTER XVIII.—SYSTEMS OF CONICS. Interpretation of $S + kS' = 0$, $S + ku^2 = 0$. Conic touching the axes. Joint equation of tangents. Confocal conics... ... 255—272

CHAPTER XIX.—ENVELOPES. Envelope of $\mu^2 P + \mu Q + R = 0$ 273—282

CHAPTER XX.—HARMONIC SECTION. Harmonic ranges and pencils. The complete quadrilateral and its harmonic properties ... ... ... .. ... ... 283—299

*Examination Paper VI.* ... ... ... ... 300

CHAPTER XXI.—CROSS RATIOS. Ranges and pencils. Pascal's Theorem. Involution ... ... ... ... 301—307

PROBLEM PAPERS .. . ... .. ... 308—322

ANSWERS ... ... . . . .. ... 323—342

INDEX .. ... .. .. ... .. .. ... 343—345

C. K. Constantinide

*PART I.*

# THE STRAIGHT LINE.

# COORDINATE GEOMETRY.

## PART I.

### THE STRAIGHT LINE.

1. **Coordinates.**—Plane Coordinate Geometry is an application of the principles of Algebra to the geometry of points, lines, and figures in a plane.

As the methods depend on the use of numbers and algebraic letters to represent lengths of lines, it must be carefully noted that in all cases some definite unit of length must be chosen in terms of which all other lengths are expressed. Thus, if a foot be chosen for the unit, the number 5 will represent 5 feet.

To fix the position of a point in a plane, we must have some points or lines, fixed in the plane, to which its position is to be referred. The simplest plan is to refer its position to two fixed straight lines intersecting at right angles.

Let $OX$, $OY$ (Fig. 1, p. 2) be these two fixed straight lines intersecting at right angles, $P$ the given point. Let $PM$ be drawn parallel to $OY$ to meet $OX$ in $M$. To get from $O$ to $P$ we have to travel a certain distance $OM$ along $OX$ and another distance $MP$ parallel to $OY$. As soon as we know the lengths of $OM$ and $MP$ we can fix the position of $P$ in the plane.

These two lengths are called the **Cartesian*** **rectangular coordinates,** or more briefly the **coordinates** of the point $P$. $OM$ is called the **abscissa,** and $MP$ the the **ordinate,** of $P$. The lines $OX$, $OY$ are called the **axes,** and $O$ is called the **origin.**

If $OM$ is $x$ units of length, and $MP$ $y$ units of length—that is to say, if the abscissa of $P$ is $x$ and the ordinate of $P$ is $y$—then $P$ is known as **the point** $(x, y)$. $OX$ is called the **axis of** $x$, and $OY$ the **axis of** $y$.

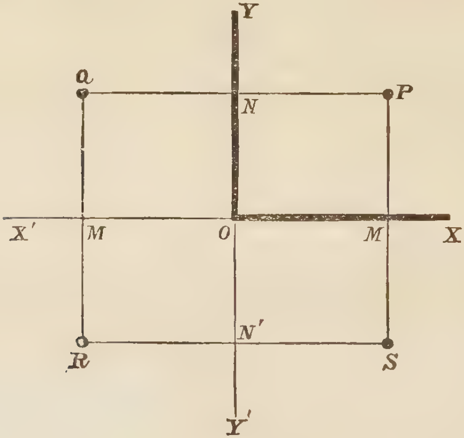



Fig. 1

Even when we know the number of units of length in $OM$ and $MP$, the position of $P$ is not completely fixed until we know whether $OM$ is drawn to the right or to the left from $O$, and whether $MP$ is drawn upwards or downwards from $M$.

We express the directions of these lines by the signs of the coordinates of $P$, and we adopt the convention which obtains in Trigonometry—that is to say, $OM$ is reckoned positive when drawn to the right from $O$, negative when drawn to the left from $O$; while $MP$ is reckoned positive when drawn upwards from $M$, negative when drawn

* The name "Cartesian" is derived from Descartes, the inventor of this system of coordinates.

downwards from $M$. In other words, the abscissæ of points to the right of $O$ are reckoned positive, those of points to the left of $O$ negative; the ordinates of points above $O$ are reckoned positive, those of points below $O$ negative.

The compartment lying between the lines $OX$ and $OY$ in the figure is known as the positive quadrant, because for all points in this compartment both abscissa and ordinate are positive.

**Oblique axes.**—In order to fix the position of a given point in a plane, it is sometimes convenient to refer the position of the point to "oblique axes"—that is, axes not inclined at right angles to each other.

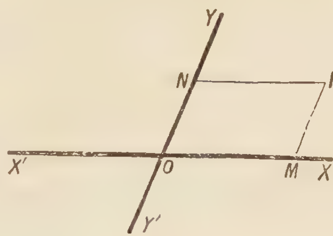



Fig. 2.

If $OX$ and $OY$ be such axes and $P$ the given point, we again draw $PM$ parallel to $OY$ to meet $OX$ in $M$. Then $OM$ is the abscissa and $MP$ the ordinate of $O$, and the same convention as to sign is maintained as in the case of rectangular axes. It is usual to denote the angle $XOY$, included between the positive directions of $OX$ and $OY$, by $\omega$, and this angle may be greater or less than a right angle.

**2. To find the distance between two points in terms of their given coordinates.**

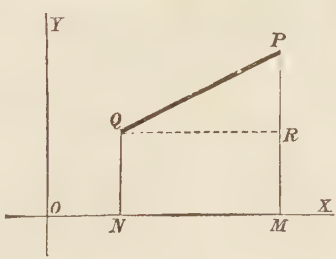



Fig. 3.

*Rectangular axes.*—Let the given points be $P$ $(x_1, y_1)$ and $Q$ $(x_2, y_2)$. Draw the ordinates $PM$ and $QN$. Through $Q$ draw $QR$ parallel to $OX$ to meet $PM$ or $PM$ produced in $R$.

Then $$PQ^2 = QR^2 + RP^2. \qquad \text{(Euc. I. 47)}$$

$$QR = OM - ON = x_1 - x_2, \quad RP = MP - NQ = y_1 - y_2;$$

$$\therefore \quad PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \quad \ldots\ldots\ldots\ldots (1).$$

COR. Taking $Q$ at the origin, $x_2 = 0$, $y_2 = 0$, and therefore $$OP^2 = x_1^2 + y_1^2.$$

*Oblique axes.*—With the same construction as before, but remembering that the ordinates are no longer perpendicular to $OX$, we have

$$PQ^2 = QR^2 + RP^2 - 2QR \,.\, RP \cos QRP.$$

As before, $QR = x_1 - x_2$ and $RP = y_1 - y_2$;

also $$\angle QRP = \pi - \omega.$$

$$\therefore \quad PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2) \cos \omega \quad \ldots\ldots\ldots (2).$$

COR. $$OP^2 = x_1^2 + y_1^2 + 2x_1y_1 \cos \omega.$$

*Note.*—When $Q$ is not in the positive quadrant either $x_2$, or $y_2$, or both, will be negative. The student should verify by drawing figures that in every case $QR = x_1 - x_2$ and $RP = y_1 - y_2$ *algebraically.*

**Exercises.**

(Exercises 1–4, 8 are for rectangular axes.)

1. Taking an inch as the unit of length, represent in a figure the positions of the following points:—

i. $(0, 0)$; ii. $(0, 1)$; iii. $(0, -1)$; iv. $(1, 0)$; v. $(-1, 0)$; vi. $(1, 1)$; vii. $(1, \frac{1}{2})$; viii. $(\frac{1}{2}, 1)$; ix. $(\frac{1}{4}, -\frac{3}{4})$; x. $(-\frac{3}{4}, -\frac{1}{4})$; xi. $(-1, \frac{5}{4})$; xii. $(-1, -1)$.

2. $P$ is the point $(a, b)$, $Q$ the point $(-a, -b)$, and $R$ the point $(b, a)$, $O$ being the origin. Prove that (i.) $O$ is the middle point of $PQ$; (ii.) $OP = OR$ and angle $XOP$ = angle $ROY$.

3. A ship is 8 miles N. and 6 miles E. of a lighthouse; another ship is 3 miles N. and 6 miles W. of the same lighthouse. Find the distance between the two ships; also the distance of the first ship from the lighthouse.

4. Find the distances of the point $(10, -18)$ from the points $(3, 6)$ and $(-5, 2)$. Show that they are equal.

5. Find the distance between the points $(4, 3)$, $(1, 2)$. $(\omega = 60°.)$

6. Find the condition that the point $(x, y)$ should be on a circle whose centre is at the origin and whose radius is $a$.

7. Find the condition that the point $(x, y)$ should lie on a circle whose centre is at the point $(d, e)$ and whose radius is $a$.

8. Show that $(71, 71)$, $(27, 9)$, $(0, 0)$, $(-13, -1)$, $(-64, 16)$ all lie on a circle whose centre is the point $(-13, 84)$ and radius 85.

**3. To find the coordinates of the point which divides the line joining two given points in a given ratio.**

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the given points, and let $R$ be the point dividing $PQ$ in the ratio $k : l$, so that

$$\frac{PR}{RQ} = \frac{k}{l}.$$

Let the coordinates of $R$ be $(x, y)$.*

(1) For *internal* division, *i.e.*, when $R$ lies in $PQ$.

Draw the ordinates $PK$, $QL$, $RM$, and through $R$ draw a line $SRT$ parallel to $OX$ to cut $PK$ and $QL$ in $S$ and $T$.

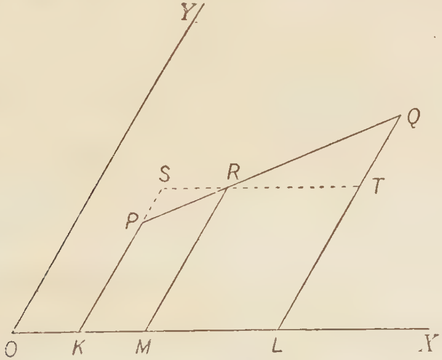



Fig. 4.

Then $$OM = OK + KM = x_1 + SR \text{ ............ } (a),$$

and $$\frac{SR}{PR} = \frac{RT}{RQ} \qquad \text{(by similar triangles)}$$

$$= \frac{SR + RT}{PR + RQ} = \frac{ST}{PQ};$$

$$\therefore \quad SR = \frac{PR}{PQ} ST = \frac{k}{k+l}(x_2 - x_1),$$

---

* Where the coordinates of a point are unknown it is usual to denote them by the plain letters $x$ and $y$; accents and suffixes being generally used when the coordinates are known.

whence, using $(a)$, we have

$$\left.\begin{aligned} x &= \frac{kx_2+lx_1}{k+l} \\ \text{Similarly} \qquad y &= \frac{ky_2+ly_1}{k+l} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(3)}.$$

COR. The coordinates of the middle point of $PQ$ are

$$\left(\frac{x_1+x_2}{2}, \ \frac{y_1+y_2}{2}\right).$$

(2) For *external* division, *i.e.* when $R$ lies in $PQ$ produced or $QP$ produced.

If $R$ divide $PQ$ *externally* in the ratio $k : l$, we have $\frac{PR}{QR} = \frac{k}{l}$.

Now $QR = -RQ$ algebraically;

$$\therefore \quad \frac{PR}{RQ} = \frac{k}{-l}.$$

Hence to get the coordinates of $R$ we have only to change the sign of $l$ in the expressions already found; hence

$$x = \frac{kx_2-lx_1}{k-l}, \quad y = \frac{ky_2-ly_1}{k-l} \quad \ldots\ldots\ldots\ldots\ldots \textbf{(4)}.$$

The student should verify this by drawing a new figure.

The above results are equally true for rectangular and for oblique axes.

*Example* (i.).—*To find the coordinates of the point dividing the line joining* $(1, -4)$, $(-3, 2)$ *externally in the ratio* $6 : 5$.

Using the above formulæ for external section, the coordinates are

$$\frac{6(-3)-5(1)}{6-5}, \quad \frac{6(2)-5(-4)}{6-5}; \quad i.e., \quad -23, \ 32.$$

*Example* (ii.).—*Assuming the result of Euclid VI. 3, prove that the internal bisectors of the angles of a triangle meet in a point.*

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle; let the lengths of the sides $BC$, $CA$, $AB$ be $a$, $b$, $c$; and let the bisectors of the angles $A$, $B$, $C$ cut the opposite sides in $D$, $E$, $F$.

Then $$BD : DC = BA : AC = c : b.$$

Hence the coordinates of $D$ are $\frac{bx_2+cx_3}{b+c}$, $\frac{by_2+cy_3}{b+c}$.

The coordinates of $A$ are $x_1$, $y_1$; hence the coordinates of the point $I$ which divides $AD$ in the ratio $b+c : a$ are

$$\frac{ax_1+bx_2+cx_3}{a+b+c}, \quad \frac{ay_1+by_2+cy_3}{a+b+c}.$$

The symmetry of this result shows that we arrive at the same point $I$ by dividing $BE$ in the ratio $c+a : b$ or $CF$ in the ratio $a+b : c$. Hence $AD$, $BE$, $CF$ all meet in $I$.

To verify these ratios geometrically remember that $I$ is the centre of the in-circle. If its radius be $r$, we have

$$\frac{AI}{ID}=\frac{\triangle AIB}{\triangle IDB}=\frac{\triangle AIC}{\triangle IDC}$$
$$=\frac{\triangle AIB+\triangle AIC}{\triangle BIC}$$
$$=\frac{\frac{1}{2}cr+\frac{1}{2}br}{\frac{1}{2}ar}=\frac{b+c}{a}.$$

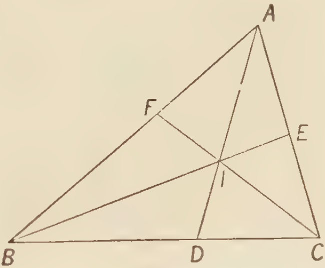


Fig. 5.

[The student should note this example carefully as an instance of symmetrical work. The proof depends upon finding a point on $AD$ whose coordinates shall be symmetrical in $a$, $b$, and $c$, in $x_1$, $x_2$, and $x_3$, and in $y_1$, $y_2$, and $y_3$; that is to say, upon so choosing $k$ and $l$ that the expression $\dfrac{k\left(\dfrac{bx_2+cx_3}{b+c}\right)+lx_1}{k+l}$ shall be symmetrical. We first choose $k$ equal to $b+c$ to clear the numerator of fractions. The coefficient of $x_2$ in the numerator being then $b$ and that of $x_3$ being $c$, we must clearly choose $l$ equal to $a$. The numerator is now symmetrical, and the denominator, being $a+b+c$, is symmetrical also. Hence the choice of the ratio $b+c : a$ is no mere guess work.]

## Exercises.

9. Find the coordinates of the point dividing the join of $(-7, 4)$, $(-6, -5)$ internally in the ratio $7 : 5$.

10. Find the coordinates of the point dividing the join of $(6, 3)$, $(-7, 2)$ externally in the ratio $1 : 5$.

11. Find the ratio in which the join of points $(6, 4,)$, $(-1, -7)$ is divided by the axis of $x$.

[Here we have to find the ratio $k : l$ so that the $y$ given by the formula is zero.]

12. The line joining $(x_1, y_1)$ and $(x_2, y_2)$ is divided into $n$ equal parts: write down the coordinates of the $r$-th point of section from the point $(x_1, y_1)$.

13. If $D$, $E$, $F$ are the middle points of the sides $BC$, $CA$, $AB$ of a triangle, and if the coordinates of $ABC$ are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, show that $AD$, $BE$, $CF$ meet in a point $G$ whose coordinates are

$$\frac{x_1+x_2+x_3}{3},\quad \frac{y_1+y_2+y_3}{3}.$$

($G$ is, of course, the centre of gravity of the triangle.)

**4. To find the area of a triangle in terms of the coordinates of its vertices.**

*Rectangular axes.*—Let the coordinates of the vertices $A$, $B$, $C$ be $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$. Draw the ordinates $AK$, $BL$, $CM$. Then

$$\triangle ABC = \text{trap}^{\text{m}}.\, BK - \text{trap}^{\text{m}}.\, BM - \text{trap}^{\text{m}}.\, CK.$$

Now $\text{trap}^{\text{m}}.\, BK$

$= \triangle BLK + \triangle BAK$

$= \frac{1}{2} LK (LB + KA)$

$= \frac{1}{2}(x_1 - x_2)(y_1 + y_2)$.

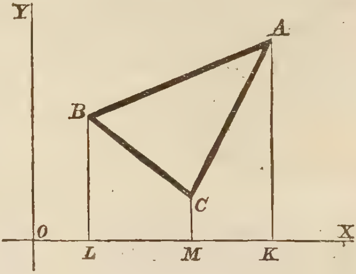



Fig. 6.

Reducing the other areas in this manner, we have

$$\triangle ABC = \tfrac{1}{2}\{x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3\} \quad \textbf{(5)}.$$

Here we have taken the vertices round the triangle in the counterclockwise order. If we take them in clockwise order, we must change the sign of the right-hand member of equation (5) to get the area.

Corollary. **The area of the triangle** $ABO = \frac{1}{2}(x_1 y_2 - x_2 y_1)$, the vertices $A$, $B$, $O$ being taken in the counterclockwise order. Hence the interpretation of (5) is that $\triangle ABC = \triangle OAB - \triangle OCB + \triangle OCA$.

*Oblique axes.*—Use the same construction. The ordinates are no longer perpendicular to $OX$; so that

$$\triangle BLK = \tfrac{1}{2} BL \times \text{perp. from } K \text{ to } BL = \tfrac{1}{2} BL \,.\, LK \sin\omega,$$

and similarly for the other areas. Then the only difference is in the introduction of the factor $\sin\omega$ in the areas throughout. Hence

$$\triangle ABC = \tfrac{1}{2}\{x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3\} \sin\omega \quad \ldots\ldots\ldots \textbf{(6)},$$

the vertices being taken in the counterclockwise order as before. Equation (5) should certainly be remembered.

*Example.*—If the coordinates of the vertices $A$, $B$, $C$, $D$ of a quadrilateral taken in the counterclockwise order are $(x_1, y_1)$, $(x_2, y_2)$, $(x_2, y_3)$, $(x_4, y_4)$ show that its area is

$$\tfrac{1}{2}\{x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_4 - x_4 y_3 + x_4 y_1 - x_1 y_4\},$$

with the addition of the factor $\sin\omega$ if the axes are oblique. (For the area is the sum of the areas of the triangles $ABC$ and $CBD$.)

**Exercises.**

14. Find the areas of the triangles formed by the following points: — (*a*) $(3, 0)$, $(-2, 1)$, $(-1, -2)$. (*b*) $(0, 0)$, $(a, b)$, $(-b, a)$. (*c*) $(p, m)$, $(0, l)$, $(p+q, n)$.

15. Find the area of the quadrilateral whose vertices are the points $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$, $(-\frac{1}{2}\sqrt{3}, \frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{2}\sqrt{3})$, $(\frac{1}{2}\sqrt{3}, -\frac{1}{2})$.

16. The vertices of a polygon of $n$ sides, taken in order are the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, ..., $(x_n, y_n)$. Show that its area is

$$\tfrac{1}{2}(x_1y_2 + x_2y_3 + \ldots + x_{n-1}y_n + x_ny_1) - \tfrac{1}{2}(y_1x_2 + y_2x_3 + \ldots + y_{n-1}x_n + y_nx_1)$$

5. **Polar coordinates.**—There is another convenient system of coordinates by which we can fix the position of a point in a plane. Take a fixed point $O$ and a fixed straight line $OX$ through $O$.

The position of any point $P$ is known when we know the length of $OP$ (usually denoted by $r$ and called the **radius vector**) and the angle which $OP$ makes with $OX$ (usually denoted by $\theta$ and called the **vectorial angle**). These are called the **polar coordinates** of $P$. The point $O$ is called the **pole,** and the line $OX$ is called the **initial line.** The angle $\theta$ is measured, as in trigonometry, in the counterclockwise direction from $OX$.

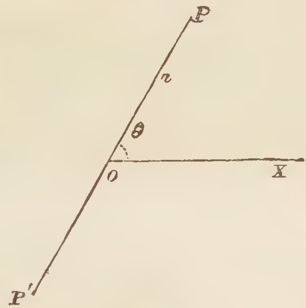



Fig. 7.

If we turn $OP$ through any number of complete revolutions, we do not alter the position of $P$.

A negative sign for $r$ would mean a distance measured in the opposite direction to $OP$, the bounding line of the vectorial angle. Hence, if we turn the revolving line through any odd number of half-revolutions, and at the same time change the sign of $r$, we do not alter the position of $P$.

Hence the coordinates of $P(r, \theta)$ in their most general form may be written $r, \theta \pm 2n\pi$ or $-r, \theta \pm (2n+1)\pi$. It is usual to choose $r$ positive and $\theta$ between 0 and $2\pi$.

**6. To change from Cartesian to polar coordinates and vice versa.**

*Rectangular axes.* — The expressions are obvious from an inspection of the accompanying figure.

Clearly

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \quad \dots\dots\dots (7).$$

$$\left.\begin{aligned} r &= \pm\sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\frac{y}{x} \end{aligned}\right\} \quad \dots (8).$$

The sign of $r$ is not really ambiguous; for $\theta$ is given in the form $n\pi+\alpha$; as soon as we have chosen any particular value for $\theta$, the sign of $r$ follows from equations (7).

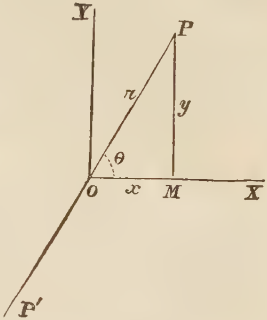



Fig. 8.

*Oblique axes.*—In the accompanying figure $OP = r$, $\angle XOP = \theta$, $\angle XMP = \omega$, $\angle OPM = \omega - \theta$.

$$\left.\begin{aligned} \therefore \quad x &= \frac{r\sin(\omega-\theta)}{\sin\omega} \\ y &= \frac{r\sin\theta}{\sin\omega} \end{aligned}\right\} \dots (9),$$

$$\left.\begin{aligned} r &= \pm\sqrt{x^2+y^2+2xy\cos\omega} \\ \theta &= \tan^{-1}\frac{y\sin\omega}{x+y\cos\omega} \end{aligned}\right\}$$

$$\dots\dots\dots (10).$$

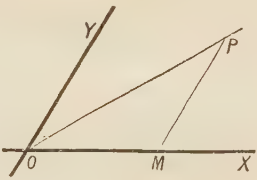



Fig. 9.

7. **To find the distance between two points in terms of their polar coordinates.**

Let $P\,(r_1, \theta_1)$ and $Q\,(r_2, \theta_2)$ be the points. Join $PQ$.

Then $\quad PQ^2 = OP^2 + OQ^2 - 2OP.OQ\cos POQ.$

$$\therefore \quad PQ^2 = r_1^{\,2} + r_2^{\,2} - 2r_1r_2\cos(\theta_2-\theta_1) \dots\dots\dots (11).$$

**Exercise.**

17. Deduce this formula from the rectangular Cartesian formula.

8. **To find the area of a triangle in terms of the polar coordinates of its vertices.**

Let $P\ (r_1, \theta_1)$, $Q\ (r_2, \theta_2)$, $R\ (r_3, \theta_3)$ be the vertices taken in the counterclockwise order, and let the point $R$ lie within the triangle $POQ$.

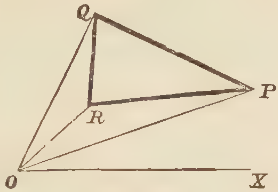


Fig. 10.

$$\triangle PQR = \triangle POQ - \triangle ROQ - \triangle POQ,$$

$$\triangle POQ = \tfrac{1}{2}\, OP.OQ \sin POQ = \tfrac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1).$$

Reducing the other areas in like manner, we have

$$\triangle POQ = \tfrac{1}{2}\{r_1 r_2 \sin(\theta_2-\theta_1) + r_2 r_3 \sin(\theta_3 - \theta_2) + r_3 r_1 \sin(\theta_1 - \theta_3)\} \quad \text{...(12)}.$$

The student should satisfy himself of the accuracy of this formula by drawing the various figures that may arise.

**Exercises.**

18. Represent in a figure, and find the rectangular coordinates of, the points whose polar coordinates are

$$(a)\ \sqrt{2},\ \frac{5\pi}{4};\quad (b)\ -\sqrt{2},\ \frac{\pi}{4};\quad (c)\ \sqrt{3},\ -\frac{\pi}{6}.$$

19. Find the polar coordinates of the points whose Cartesian coordinates are

$$(a)\ -\sqrt{2},\ -\sqrt{2};\quad (b)\ \tfrac{1}{2}\sqrt{3},\ -\tfrac{3}{2};\quad (c)\ -5,\ 0.$$

20. Find the distances between

$(a)$ $(1\tfrac{1}{6}, \pi)$ and $(2\sqrt{3}, \tfrac{1}{3}\pi)$; $(b)$ $(2\sqrt{2}, \tfrac{1}{12}\pi)$ and $(2, \tfrac{1}{3}\pi)$.

21. Find the areas of the triangles formed by

$(a)$ $(1, \tfrac{1}{8}\pi)$, $(1, \tfrac{5}{8}\pi)$, $(\sqrt{2}, \tfrac{3}{8}\pi)$; $(b)$ $(3, \tfrac{1}{6}\pi)$, $(2, \tfrac{1}{2}\pi)$, $(1, \tfrac{5}{6}\pi)$.

22. Deduce formula (12) from the rectangular Cartesian formula.

23. If $(r_1, \theta_1)$, $(r_2, \theta_2)$, $(r_3, \theta_3)$, $(r_4, \theta_4)$ be the polar coordinates of the vertices $A$, $B$, $C$, $D$ of a quadrilateral taken in order counterclockwise, show that its area is

$$\tfrac{1}{2}\{r_1 r_2 \sin(\theta_2 - \theta_1) + r_2 r_3 \sin(\theta_3 - \theta_2) + r_3 r_4 \sin(\theta_4 - \theta_3) + r_4 r_1 \sin(\theta_1 - \theta_4\}.$$

9. **Loci and equations.**

**A locus is the path of a point which moves according to some fixed rule.**

For instance, the locus of a point which moves so as always to be at a fixed distance from a given point is a circle. When a point is not restricted to one particular position, but moves along a line, its coordinates are said to be current.

If a point moves according to some fixed rule, its coordinates will always satisfy some corresponding algebraical relation. Thus, if the point moves so as always to be at a distance $a$ from the origin $O$, its rectangular Cartesian coordinates will always satisfy the equation

$$x^2+y^2=a^2. \qquad (\S\ 2)$$

Corresponding, therefore, to the curve* traced by a point moving according to any fixed rule we have an invariable algebraical relation satisfied by the coordinates of every point on the curve, and called the **equation of the curve**; and, conversely, corresponding to every algebraical equation connecting the coordinates of a moving point we have a curve on which the point must lie so long as its coordinates satisfy the given equation, and called the **locus of the equation.**

Thus in the example given above the equation of the circle whose centre is $O$ and radius $a$ is $x^2+y^2=a^2$; and the locus of the equation $x^2+y^2=a^2$ is a circle whose centre is $O$ and radius $a$.

We have considered rectangular Cartesian coordinates for simplicity, but obviously a curve will also have its equation in oblique Cartesian and in polar coordinates. With this introduction the following definitions should be clear :—

The **equation of a curve** is the algebraical relation which is satisfied by the coordinates of every point on the curve.

Conversely, the **locus of an equation** is the curve the coordinates of every point on which satisfy that equation. The curve must contain *all* the points satisfying this

* The student must accustom himself to the use of the word "curve" to denote any line, or set of lines (straight or curved), obeying some fixed rule.

condition; so that no point off the curve satisfies the condition.

The coordinates of the point or points of intersection of two curves whose equations are given must satisfy both equations, since any point of intersection is on both curves. Hence the coordinates required are found by solving simultaneously the two equations for $x$ and $y$ or $r$ and $\theta$, as the case may be.

The above definitions and introductory paragraphs are of prime importance, and must be thoroughly mastered before proceeding further.

To obtain an idea of the locus of a given equation, we can take a succession of given values for $x$, and solve the equation for the corresponding values of $y$. We thus obtain a number of different points on the locus, which we can plot on a diagram (preferably with the aid of paper ruled in squares). This will serve as a guide to the locus.

*Example.—Trace the locus* $x-y=0$.

By the above method we find that the points $(0, 0)$, $(1, 1)$, $(2, 2)$, $(-1, -1)$, $(-2, -2)$ are on the locus. These points *suggest* the bisector of the angle $XOY$. We have yet to prove that the coordinates of *every* point on this bisector satisfy the given equation. This, however, is obvious from Euclid I. 6; hence the bisector is the locus required.

**Exercises.**

24. Show that the equations of the axes of $x$ and $y$ respectively are $y=0$ and $x=0$.

25. Show that the locus of the equation $r=a$ is a circle, and the locus of the equation $\theta=\alpha$ is a straight line.

26. Show that the lines whose equations are $x-y=5$ and $x+y=7$ intersect in the point $(6, 1)$.

27. Find the locus of a point, having given that the sum of the squares of its distances from the axes is equal to 2.

28 Find the locus of a point the square of whose distance from the point $(0, 1)$ is equal to unity.

29. Find the equation of the locus of which every ordinate is greater than the corresponding abscissa by a given distance.

30. Trace the locus of each of the following equations:—

(*a*) $2x-y=0$. (*b*) $x^2+y^2=16$. (*c*) $y^2=4x^2$. (*d*) $y^2=4$.
(*e*) $x^2=y$. (*f*) $xy=0$. (*g*) $3x+2y=6$. (*h*) $x^2=xy$.

**10. To find the equation of a straight line,**

(A) **Parallel to an axis.**—The equation of a straight line parallel to axis of $x$ is $y=b$, where $b$ is the ordinate of any point on the line. Similarly $x=a$ is the equation of a straight line parallel to the axis of $y$, where $a$ is the abscissa of any point on the line.

(B) **Given the direction of the straight line and the coordinates of one point on it.**

*Rectangular axes.*—Let $Q(x_1, y_1)$ be the given point, and let the straight line make an angle $\theta$ (measured counterclockwise) with the axis of $x$. Let $P(x, y)$ be any point on the straight line. Draw the ordinates $QN$, $PM$, and draw $QR$ parallel to $OX$ to cut $PM$ or $PM$ produced in $R$.

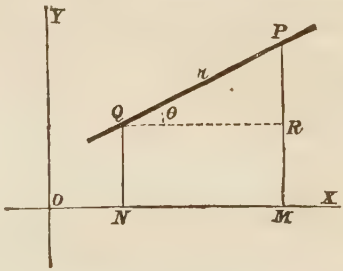



Fig. 11.

The geometrical condition satisfied by $P$ is

$$RP = QR\tan\theta.$$

The algebraical expression of this is

$$\left.\begin{aligned} y-y_1 &= (x-x_1)\tan\theta \\ \text{or}\qquad \frac{x-x_1}{\cos\theta} &= \frac{y-y_1}{\sin\theta} \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(\mathbf{13}),$$

which is therefore the equation required. This equation is usually written in the form

$$y-y_1 = m(x-x_1)\quad\ldots\ldots\ldots\ldots(\mathbf{14}),$$

in which case the inclination $\theta$ is $\tan^{-1}m$.

If the straight line cuts the axis of $y$ at a height $b$ above the origin, we can take $Q$ at the point $(0, b)$, and the equation is

$$y = mx+b\quad\ldots\ldots\ldots\ldots(\mathbf{15}).$$

And, if the straight line passes through the origin, its equation is

$$y = mx\quad\ldots\ldots\ldots\ldots(\mathbf{16}).$$

COR. The coordinates of $P$ are

$$x_1 + r\cos\theta,\quad y_1 + r\sin\theta.$$

*Oblique axes.*—With the same construction, we have

$\angle PQR = \theta$,

$\angle QPR = \angle YOX - \angle PQR$
$= \omega - \theta.$

$$\frac{PR}{QR} = \frac{\sin\theta}{\sin(\omega-\theta)};$$

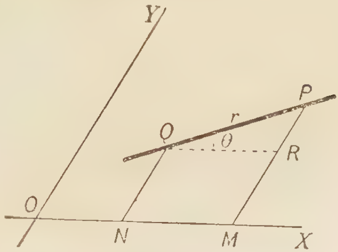


Fig. 12.

so that we have the equations (13–16) as before, except that $m$ is now $\dfrac{\sin\theta}{\sin(\omega-\theta)}$, whence it can be shown that the inclination

$$\tan\theta = \frac{m\sin\omega}{1+m\cos\omega} \quad \text{.............. (17).}$$

*Example.—The axes being inclined at 45°, find the angle the line $y = \frac{1}{2}(\sqrt{6}+\sqrt{2})\,x$ makes with the axis of $x$.*

Here $\qquad \omega = 45°$ and $m = \frac{1}{2}(\sqrt{6}+\sqrt{2})$;

$$\therefore \quad \tan\theta = \frac{m\sin\omega}{1+m\cos\omega} = \frac{m\dfrac{1}{\sqrt{2}}}{1+\dfrac{m}{\sqrt{2}}} = \frac{m}{m+\sqrt{2}}.$$

Thus $\quad \tan\theta = \dfrac{\frac{1}{2}(\sqrt{6}+\sqrt{2})}{\frac{1}{2}(\sqrt{6}+\sqrt{2})+\sqrt{2}} = \dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+3\sqrt{2}} = \dfrac{1}{\sqrt{3}}.$

$$\therefore \quad \theta = 30°.$$

(C) **Given the coordinates of two points on the straight line.**

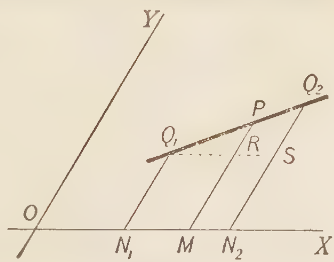


Fig. 13.

Let $Q_1\ (x_1, y_1)$ and $Q_2\,(x_2, y_2)$ be the points, and let $P\,(x, y)$ be any point on the line.

Draw the ordinates $Q_1N_1$, $Q_2N_2$, $PM$, and draw $Q_1RS$ parallel to $OX$ to cut $PM$ in $R$ and $Q_2N_2$ in $S$.

The geometrical condition satisfied by $P$ is

$$\frac{Q_1R}{Q_1S} = \frac{RP}{SQ_2}.$$

The algebraical expression of this is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(18)},$$

which is the equation required, and holds equally good for rectangular and for oblique axes.

*Example.—Find the equation of the line joining* $(-3, 7)$, $(1, -2\frac{1}{2})$.

The equation required is

$$\frac{y-7}{-2\frac{1}{2}-7} = \frac{x-(-3)}{1-(-3)}, \quad \text{or} \quad \frac{y-7}{-\frac{19}{2}} = \frac{x+3}{4},$$

*i.e.*, on reduction, $19x + 8y + 1 = 0$.

(D) **Given the lengths of the intercepts on the axes.**

This is a particular case of (C), but it is better to give an independent proof.

*Rectangular axes.*—Let the intercepts on the axes of $x$ and $y$ be $OA$ $(=a)$ and $OB$ $(=b)$. Let $P$ $(x, y)$ be any point on the line.

The geometrical condition satisfied by $P$ is

$$\triangle OBP + \triangle OAP = \triangle OAB.$$

The algebraical expression of this is

$$\tfrac{1}{2}bx + \tfrac{1}{2}ay = \tfrac{1}{2}ab$$

or

$$\frac{x}{a} + \frac{y}{b} = \mathbf{1} \quad \textbf{(19)}.$$

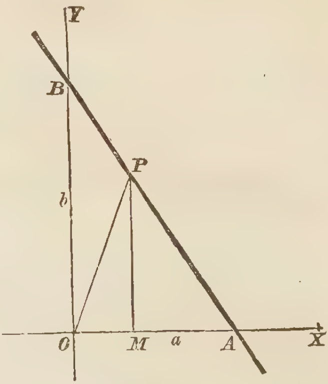


Fig. 14.

This equation holds equally good for oblique axes, the only difference in the proof being that the factor sin $\omega$ occurs throughout in the expressions for the areas and therefore cancels out finally.

(E) **Perpendicular form—given the length ($p$) of the perpendicular from the origin on the straight line, and the angle ($\alpha$) made by this perpendicular with the axis of $x$.**

[$\alpha$ is measured in the counterclockwise direction.]

Note that, as already explained in polar coordinates, the perpendicular $p$ is necessarily positive; for, if it were negative, $\alpha$ would be the angle made with the axis of $x$ by the direction of the perpendicular reversed (that is, produced backwards through the origin) and not by the perpendicular itself.

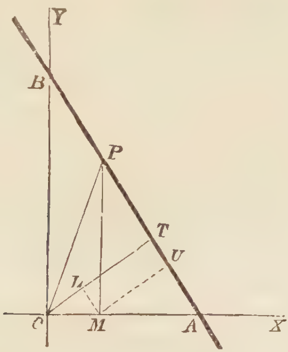



Fig. 15.

*Rectangular axes.*—Let the straight line cut $OX$ and $OY$ in $A$ and $B$.

Let $OT$ be the perpendicular from the origin on the straight line.

Then $$OT = p, \ \angle XOT = \alpha.$$

Let $P(x, y)$ be any point on the line.

Draw the ordinate $PM$, and draw $ML$ perpendicular to $OT$ and $MU$ perpendicular to $AB$.

Then $OT = OL + MU = OM \cos AOT + MP \sin MPU$.

Now $\angle MPU = \angle OBT = 90° - \angle BOT = \angle AOT$.

Therefore the algebraical equivalent is

$$x \cos \alpha + y \sin \alpha = p \quad \ldots\ldots\ldots\ldots \quad (20).$$

*Oblique axes.*—With the same construction we have, as before, $$OT = OM \cos AOT + MP \cos BOT,$$
and the algebraical expression of this is
$$x \cos \alpha + y \cos (\omega - \alpha) = p \quad \ldots\ldots\ldots \quad (21).$$

(F) **The polar equation of a straight line (general form).**

Let $OT$ be the perpendicular from the pole on the straight line, and let the polar coordinates of $T$, as before, be $(p, \alpha)$. Let $P(r, \theta)$ be any point on the straight line.

The geometrical condition satisfied by $P$ is

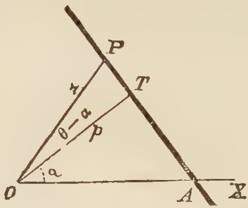



Fig. 16.

$$OP \cos TOP = OT.$$

Now $$\angle TOP = \angle AOP - \angle AOT = \theta - \alpha.$$

Therefore the algebraical expression of this is
$$r \cos (\theta - \alpha) = p \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22).$$

**Exercises.**

31. Prove equation (18) by using the fact that the area of the $\triangle PQ_1Q_2$ is zero if $P$ lies on the straight line $Q_1Q_2$.

32. Deduce the Cartesian form (20) from the polar form (22) and *vice versa*.

(G) **Polar equation of a straight line passing through two given points.**

Let $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ be the given points, $P(r, \theta)$ any point on $AB$.

The geometrical condition satisfied by $P$ is that the area of the triangle $ABP$ is zero. Hence (§ 8) the equation of the straight line is

$$\left.\begin{aligned} r_1r_2 \sin (\theta_2 - \theta_1) + r_2r \sin (\theta - \theta_2) + rr_1 \sin (\theta_1 - \theta) &= 0 \\ \text{or } \frac{1}{r} \sin (\theta_2 - \theta_1) + \frac{1}{r_1} \sin (\theta - \theta_2) + \frac{1}{r_2} \sin (\theta_1 - \theta) &= 0 \end{aligned}\right\}$$
$$\ldots\ldots\ldots \quad (23).$$

**Exercises.**

33. Find the lines through the point (0, 2) making angles $\frac{1}{3}\pi$ and $\frac{2}{3}\pi$ with the axis of $x$. Also the lines parallel to them cutting the axis of $y$ at a distance 2 below the origin. Find also their points of intersection with the axis of $x$.

34. What are the inclinations to the axis of $x$ of the lines

$$y = \tfrac{1}{3}x\sqrt{3} + 3 \text{ and } y = \sqrt{3}x + 3\,?$$

Show that the line $y = x + 3$ bisects the angle between them.

35. Show that $(-1, 3)$, $(3, 2)$, $(11, 0)$ lie in a straight line.

36. If any ordinate parallel to $OY$ cut the straight lines $y = mx$ and $y = mx + b$, the portion intercepted on it is of constant length.

37. Find the equations of the tangents to a circle, whose centre is the origin and radius $\sqrt{2}$, at the two extremities of a diameter making an angle 45° with the axis of $x$.

38. Find the equation of the straight line bisecting the line joining the points (5, 3) and (4, 4), and making an angle of 45° with the axis of $x$.

39. The vertices of a triangle are the points (0, 0), (2, 4), and $(-6, 4)$. Find the equations of its sides.

40. Find the equation of a straight line passing through the point (2, 2), such that the sum of its intercepts on the axis $= 9$.

41. Find the equation of the line through the point $Q$ (2, 3) making an angle of 45° with the axis of $x$, and determine the length intercepted on it between the point $Q$ and the line $x + y + 1 = 0$.

42. Find the ratio in which the join of the points (6, 4), $(-1, -7)$ is divided by the line $y + 4x = 0$.

43. If $\omega = 45°$, show that the line $y = x$ makes an angle $22\frac{1}{2}°$ with the axis $OX$; and, in general, show, by means of the formula, that the line $y = x$ makes an angle $\frac{1}{2}\omega$ with $OX$.

44. If $\tan \omega = \frac{4}{3}$ ($\omega$ being $< \frac{1}{2}\pi$), find the angle that the line $y = 5x$ makes with $OX$.

## 11. General form of the equation of the straight line referred to Cartesian coordinates.

All the forms in which we have found the equation of the straight line referred to Cartesian coordinates are of the first degree in $x$ and $y$. The most general form of any equation of the first degree in $x$ and $y$ is

$$Ax + By + C = 0.$$

This equation, though apparently involving three constants, $A$, $B$, and $C$, in reality only involves two—namely, the ratios $A/C$ and $B/C$.

**12. To prove that the general equation of the first degree in $x$ and $y$ represents a straight line.**

Let $Q_1(x_1, y_1)$ and $Q_2(x_2, y_2)$ be the two selected points, $P(x_3, y_3)$ the third point on the locus $Ax+By+C=0$.

Draw the ordinates $Q_1N_1$, $Q_2N_2$, $PM$, and draw $QRS$ parallel to $OX$ to cut $PM$ in $R$ and $Q_2N_2$ in $S$. Then, since the coordinates of the three points satisfy the equation of the locus, we have

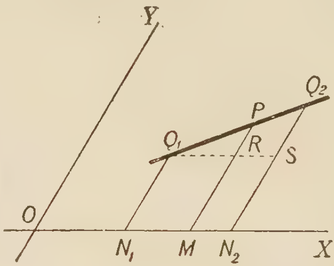



Fig. 17.

$$Ax_1+By_1+C=0 \quad\ldots\ldots\ldots\ldots(1),$$
$$Ax_2+By_2+C=0 \quad\ldots\ldots\ldots\ldots(2),$$
$$Ax_3+By_3+C=0 \quad\ldots\ldots\ldots\ldots(3).$$

From (1) and (2), $A(x_2-x_1)+B(y_2-y_1)=0$.

From (1) and (3), $A(x_3-x_1)+B(y_3-y_1)=0$.

Hence $\dfrac{x_3-x_1}{x_2-x_1}=\dfrac{y_3-y_1}{y_2-y_1}$; that is, $\dfrac{Q_1R}{Q_1S}=\dfrac{RP}{SQ_2}$.

Hence $P$ lies on the line joining $Q_1Q_2$, and the theorem is proved.

*Alternative Proof.*—The general equation of the first degree may be written $y=-Ax/B-C/B$, which is of the form $y=mx+b$. Now we have found that this last equation is satisfied by the coordinates of every point on a certain straight line which cuts the axis of $y$ at a distance $b$ from the origin and is inclined at an angle $\tan^{-1}m$ (for rectangular axes) or $\tan^{-1}m\sin\omega/(1+m\cos\omega)$ (for oblique axes) to the axis of $x$.

Further, the equation is satisfied by the coordinates of no other points. For let $x_1, y_1$ be the coordinates of a point $Q$ above the line referred to, and let $y_1'$ be the ordinate of the point $P$ on this line whose abcissa is the same as that of $Q$, viz. $x_1$.

Then $y_1' = mx_1 + b$, since $(x_1, y_1')$ is on the line.
But $y_1 > y_1'$, since $Q$ is above $P$.

$$\therefore \quad y_1 > mx_1 + b.$$

Similarly, if $Q$ be below the line in question,

$$y_1 < mx_1 + b.$$

Hence the equation $y = mx + b$ is satisfied by the coordinate of every point on a certain straight line, but of no other point; hence it is the equation of a straight line.

13. It follows from the reasoning of the last paragraph that the points $(x_1, y_1)$ and $(x_2, y_2)$ are on the same or on opposite sides of the straight line $y = mx + b$ according as the expressions $(y_1 - mx_1 - b)$ and $(y_2 - mx_2 - b)$ have like or unlike signs; and, since the general equation $Ax + By + C = 0$ is only the equation $y = mx + b$ multiplied throughout by a constant and transformed, it also follows that the points $(x_1, y_1)$ and $(x_2, y_2)$ are on the same or on opposite sides of the straight line $Ax + By + C = 0$ according as the expressions

$$Ax_1 + By_1 + C \quad \text{and} \quad Ax_2 + By_2 + C$$

(that is, the respective results of substituting the coordinates $x_1$, $y_1$ and $x_2$, $y_2$, in the left-hand member of the equation of the straight line) have like or unlike signs.

14. The equation $Ax + By + C = 0$ may be written in the forms

$$y = -\frac{A}{B}x - \frac{C}{B} \quad \text{and} \quad \frac{x}{\frac{-C}{A}} + \frac{y}{\frac{-C}{B}} = 1,$$

from which, by comparison with the forms $y = mx + b$ and $x/a + y/b = 1$, we see that the inclination of the straight line given by the general equation to the axis of $x$ is $\tan^{-1}(-A/B)$ if the axes are rectangular and

$$\tan^{-1}\left(\frac{-A \sin \omega}{B - A \cos \omega}\right)$$

if the axes are oblique; and that the intercepts on the axes of $x$ and $y$ respectively are $-C/A$ and $-C/B$.

15. The reduction of the equation $Ax+By+C=0$ to the "perpendicular" form $x\cos\alpha+y\sin\alpha=p$ (the axes being rectangular) requires a little more care. Since the sum of the squares of the coefficients of $x$ and $y$ in the reduced equation has to be unity, we must reduce by dividing by $\sqrt{A^2+B^2}$. The equation then takes the form

$$-\frac{A}{\sqrt{A^2+B^2}}x-\frac{B}{\sqrt{A^2+B^2}}y=\frac{C}{\sqrt{A^2+B^2}} \quad \ldots\ldots (1).$$

Remembering that $p$ is necessarily positive [§ 10 (E)], we see that equation (1) is the correct reduced form when $C$ is positive, and equation (1) with the signs changed all through is the correct reduced form when $C$ is negative.

The perpendicular from the origin is $\dfrac{C}{\sqrt{A^2+B^2}}$ or $\dfrac{-C}{\sqrt{A^2+B^2}}$ according as $C$ is positive or negative.

For oblique axes, comparing the equation $Ax+By+C=0$ with $x\cos\alpha+y\cos(\omega-\alpha)=p$, we have

$$-pA/C=\cos\alpha,$$

$$-pB/C=\cos(\omega-\alpha)=\cos\omega\cos\alpha+\sin\omega\sin\alpha.$$

$$\therefore\quad -pB/C+pA\cos\omega/C=\sin\omega\sin\alpha$$

and
$$-pA\sin\omega/C=\sin\omega\cos\alpha.$$

Squaring and adding,

$$\frac{p^2}{C^2}\left\{(A\cos\omega-A)^2+A^2\sin^2\omega\right\}=\sin^2\omega,$$

$$\therefore\quad p=\pm\frac{C\sin\omega}{\sqrt{A^2+B^2-2AB\cos\omega}},$$

and the reduced form of the equation is

$$-\frac{A\sin\omega}{\sqrt{(A^2+B^2-2AB\cos\omega)}}x-\frac{B\sin\omega}{\sqrt{(A^2+B^2-2AB\cos\omega)}}y$$
$$=\frac{C\sin\omega}{\sqrt{(A^2+B^2-2AB\cos\omega)}} \quad \ldots (2),$$

if $C$ is positive, and the same equation with all the signs changed if $C$ is negative.

**16. To find the perpendicular distance of the point $(x', y')$ from the straight line $x \cos \alpha + y \sin \alpha = p$, the axes being rectangular.**

Let $AB$ be the straight line, $P$ the point $(x', y')$, $d$ the required length of the perpendicular $NP$, $P$ being assumed to lie on the opposite side of $AB$ from $O$.

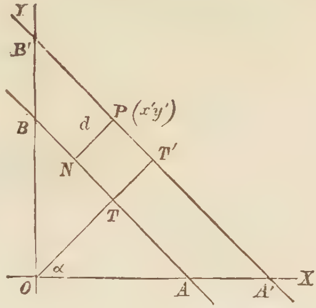


Fig. 18.

Draw $A'PB'$ parallel to $AB$ through $P$, and draw the common perpendicular $OTT'$ from $O$ on $AB$ and $A'B'$.

Then $OT = p$, and angle $XOT = \alpha$ [§ 10 (E)]. The perpendicular from $O$ on $A'B'$ is of length $OT'$, and makes an angle $\alpha$ with $OX$; hence the equation of $A'B'$ is

$$x \cos \alpha + y \sin \alpha = OT' = p + d. \qquad [\S\, 10\ (\mathrm{E})]$$

The coordinates $(x', y')$ of $P$ satisfy the equation of $A'B'$; $\quad \therefore \quad x' \cos \alpha + y' \sin \alpha = p + d$:

$$\therefore \quad \boldsymbol{d = x' \cos \alpha + y' \sin \alpha - p} \quad \ldots\ldots\ldots\ldots\mathbf{(24)};$$

so that the length of the perpendicular is the result of substituting the coordinates of $P$ in the expression $x \cos \alpha + y \sin \alpha - p$.

To verify geometrically, draw $PM$ the ordinate of $P$, and $MU$ perpendicular to $OT'$, and note that equation (24), interpreted geometrically, becomes $NP = OU + UT' - OT$.

### Exercises.

45. Find whether the points (1, 1) and (2, 2) lie on the same or on opposite sides of the line $x - 3y + 5 = 0$.

46. Show that the origin and the three points $(1, 1)$, $(0, \frac{5}{8})$, $(\frac{9}{4}, -3)$ are in the four different compartments made by the lines

$$3x + 2y = 1 \quad \text{and} \quad 5x + 3y = 2.$$

47. Reduce (*a*) the equation $3x + 4y - 10 = 0$ to the perpendicular form, the axes being rectangular; (*b*) the equation $x + y + 4 = 0$ to the perpendicular form, the axes including an angle of 60°.

48. **Prove that, for oblique axes, the length of the perpendicular from $(x', y')$ on $x \cos \alpha + y \cos (\omega - \alpha) = p$ is**

$$\boldsymbol{x' \cos \alpha + y' \cos (\omega - \alpha) - p}.$$

17. By reducing the general equation $Ax+By+C=0$ to the perpendicular form, we see that (without regard to sign) the length of the perpendicular from $(x', y')$ on $Ax+By+C=0$ is

$$\frac{Ax'+By'+C}{\sqrt{A^2+B^2}} \quad \text{.....................} \quad (25)$$

for rectangular axes, and

$$\frac{Ax'+By'+C}{\sqrt{(A^2+B^2-2AB\cos\omega)}}\sin\omega \quad \text{......} \quad (26)$$

for oblique axes.

18. **Sign of the expression for the perpendicular.** So far we have dealt with the length of the perpendicular. We must now consider its sign. It is evident that this cannot be found by the mere substitution of the coordinates $x', y'$ in the expression $Ax+By+C$, since the general equation may be equally well written in either of the forms $Ax+By+C=0$ or $-Ax-By-C=0$. It being therefore impossible to assign any absolute sign to the perpendicular, we fix its direction by finding on which side of the given straight line the point $(x', y')$ lies. The simplest plan is to find whether $(x', y')$ lies on the same side as the origin or on the opposite side, and this is done by substituting in turn the coordinates of $(x', y')$ and the origin in the expression $Ax+By+C$, and observing whether the results have like or unlike signs (§ 13).

*Example.*—*Find the perpendicular distance of the point $(a, b)$ from the line $x/a+y/b=1$, the axes being rectangular.*

The required distance is

$$d = \frac{\frac{a}{a}+\frac{b}{b}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} = \frac{2-1}{\sqrt{\frac{a^2+b^2}{a^2b^2}}}$$

$$= \frac{ab}{\sqrt{a^2+b^2}}.$$

**19. To find the angle between two straight lines whose equations are given.**

*Rectangular axes.*—Let $y = m_1x + b_1$ and $y = m_2x + b_2$ be the equations of the straight lines, $\theta_1$ and $\theta_2$ their inclinations to the axis of $x$, $\phi$ the angle between them.

Then $\quad \phi = \theta_1 - \theta_2, \quad \tan\theta_1 = m_1, \quad \tan\theta_2 = m_2;$

$$\therefore \quad \tan\phi = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2};$$

$$\therefore \quad \boldsymbol{\tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{(27)}.$$

[Note that, if we reverse the order of the lines, we get for $\tan\phi$ the same value with the opposite sign; that is, the tangent of the supplement of $\phi$.]

*Oblique axes.*—Here

$$\tan\theta_1 = \frac{m_1 \sin\omega}{1 + m_1 \cos\omega}, \quad \tan\theta_2 = \frac{m_2 \sin\omega}{1 + m_2 \cos\omega},$$

whence

$$\boldsymbol{\tan\phi = \frac{(m_1 - m_2)\sin\omega}{1 + (m_1 + m_2)\cos\omega + m_1 m_2}} \quad \ldots\ldots\mathbf{(28)}.$$

**Exercise.**

49. By reducing the general equation of the straight line to the "tangent" form $y = mx + b$, show that the angle between the straight lines whose equations are $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ is

$$\tan^{-1}\frac{A_2B_1 - A_1B_2}{A_1A_2 + B_1B_2} \quad \text{or} \quad \tan^{-1}\frac{(A_2B_1 - A_1B_2)\sin\omega}{A_1A_2 - (A_1B_2 + A_2B_1)\cos\omega + B_1B_2},$$

according as the axes are rectangular or oblique.

The student is advised not to commit this last result to memory. In working numerical examples in practice, it is best to reduce first to the "tangent" form, and then apply formula (27) or formula (28), as the case may be.

Two results, however, should be remembered:

(*a*) **The condition of parallelism** ($\tan\phi = 0$) is

$$\boldsymbol{m_1 = m_2} \quad \text{or} \quad \boldsymbol{A_1/A_2 = B_1/B_2} \quad \ldots\ldots\ldots \mathbf{(29)}$$

for both rectangular and oblique axes.

(*b*) **The condition of perpendicularity** ($\tan\phi = \infty$) for rectangular axes is

$$\boldsymbol{m_1 m_2 = -1} \quad \text{or} \quad \boldsymbol{A_1A_2 = -B_1B_2} \quad \ldots\ldots \mathbf{(30)}.$$

Hence the equation of any straight line parallel to $Ax+By+C=0$ may be found by a suitable alteration of the constant term only (for the ratio $A:B$ remains unaltered). Any particular line of this series will be defined by a second condition besides that of parallelism to $Ax+By+C=0$, and this second condition will determine the value to be given to the constant term to obtain the equation of the particular line in question.

If the axes are rectangular, the equation of any straight line perpendicular to $Ax+By+C=0$ may be found by interchanging the coefficients of $x$ and $y$ and changing the sign of one of them, leaving the constant term to be determined as before.

### Exercises.

[Nos. 50–60 are for rectangular axes.]

50. Find the perpendicular distance of the point $(b, a)$ from the line $x/a+y/b=2$.

51. Show that the origin is equidistant from the three straight lines $4x+3y+10=0$, $5x-12y+26=0$, $7x+24y=50$.

52. Find the coordinates of the vertices of the triangle whose sides are $3x+y=7$, $3y-x=1$, and $7y+x+11=0$; and find the lengths of the perpendiculars from these vertices on the opposite sides.

53. Find the angle between the straight lines $y-\sqrt{3}x-5=0$ and $\sqrt{3}y-x+6=0$.

54. Find the angle between the straight lines $2x-y+1=0$ and $11x-(8+5\sqrt{3})y+5\sqrt{3}=0$.

55. Find the angle between the straight lines $y=mx+b$ and $(m-1)x-(m+1)y=(m^2-1)b$.

56. Find the equations of the straight lines through $(-1, -3)$ parallel and perpendicular to $x+7y=2$.

57. Find the equation of the straight line through the origin perpendicular to the line joining the points $(3, -6)$ and $(4, 5)$.

58. Find the equations of the lines through the origin making angles of 60° with the line $x+y\sqrt{3}+3\sqrt{3}=0$; also the coordinates of the points where they meet the line.

59. Find the point on the axis of $x$ whose perpendicular distance from the line $3x+4y=12$ is 4, and which is on the side of the line remote from the origin.

60. Find the coordinates of the foot of the perpendicular from the point $(h, k)$ on the line $y=mx+b$.

61. Find the condition that the lines $2x+3y=4$, $3x-ky=2$ be parallel, the angle between the axes being 30°.

62. What is the value of $k$ if $3x+4y=5$ and $4x+ky=3$ are perpendicular, the angle between the axes being $\tan^{-1}\frac{5}{12}$?

63. Find the angle between the straight lines whose equations are $y=x\sqrt{3}+5$ and $y=\frac{1}{3}x\sqrt{3}+2$, the axes being rectangular.

64. Find the angle between the straight lines whose equations are $2x-y+7=0$ and $3x+6y-8=0$, the axes being rectangular.

## 20. Coordinates of point of intersection of two straight lines.

We have already seen (Art. 9) that the coordinates of the point of intersection of the straight lines $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$ are obtained by solving these two equations simultaneously for $x$ and $y$. We find by solving that these coordinates are

$$\frac{B_1C_2-B_2C_1}{A_1B_2-A_2B_1},\quad \frac{C_1A_2-C_2A_1}{A_1B_2-A_2B_1}.$$

## 21. Condition of concurrency of three straight lines.

The condition that the three straight lines whose equations are

$$A_1x+B_1y+C_1=0 \qquad (1),$$
$$A_2x+B_2y+C_2=0 \qquad (2),$$
$$A_3x+B_3y+C_3=0 \qquad (3)$$

should be concurrent is that the coordinates of the point of intersection of (1) and (2) should satisfy the equation of 3; that is

$$A_3(B_1C_2-B_2C_1)+B_3(C_1A_2-C_2A_1)+C_3(A_1B_2-A_2B_1)=0.$$

The student is advised not to commit this result and that of § 20 to memory; it is sufficient to grasp and retain the principle.

The student acquainted with the theory of determinants should observe equation ($a$). The condition of concurrency requires that equations (1), (2), and (3) should have a common solution in $x$ and $y$. Eliminating $x$ and $y$ from the equations, the condition for this is

$$\begin{vmatrix} A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \\ A_3, & B_3, & C_3 \end{vmatrix} = 0.$$

The above equation is simply the expansion of this condition.

### Exercise.

65. Find the coordinates of the point of intersection of the straight lines $4x+3y=10$ and $3x+5y=13$.

**22. Equation of a straight line passing through the point of intersection of two given straight lines.**

The equation

$$A_1x+B_1y+C_1+k(A_2x+B_2y+C_2)=0 \;\ldots\; (31)$$

represents, for all values of $k$, some straight line passing through the point of intersection of $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$.

For equation (31) is of the first degree in $x$ and $y$, and therefore represents some straight line. Further, since the coordinates of the point of intersection of $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$ satisfy the equations of both these lines, they also satisfy equation (31). Hence that equation represents a straight line through the point of intersection of the two lines.

By giving a suitable value to $k$, we can make the straight line represented by equation (31) satisfy any second condition.

*Example.—Find the equation of the straight line parallel to the axis of y drawn through the point of intersection of* $x-7y+5=0$ *and* $3x+y-7=0$.

The equation of any straight line through the point of intersection is of the form

$$x-7y+5+k(3x+y-7)=0 \quad \text{or} \quad (1+3k)x+(k-7)y+5-7k=0.$$

If this line is parallel to the axis of $y$, the coefficient of $y$ is zero; hence $k=7$, and the equation becomes $x-2=0$.

**Exercises.**

66. Find the equation of the straight line drawn through the same point of intersection parallel to the axis of $x$.

67. Verify the results of the two preceding problems by finding the coordinates of the point of intersection of $x-7y+5=0$ and $3x+y-7=0$.

23. Following out the reasoning of the last article, the condition of concurrency may be presented in a form which is sometimes useful. If we can find three constants $l$, $m$, $n$ such that the expression

$$l(A_1x+B_1y+C_1)+m(A_2x+B_2y+C_2)$$
$$+n(A_3x+B_3y+C_3) \;\ldots\ldots\; (32)$$

is *identically* equal to zero [*i.e.* is zero for all values of $x$ and $y$, which requires that the coefficient of $x$, the coefficient of $y$, and the constant term in (32) should each be zero when the appropriate values are given to $l$, $m$, and $n$], then the three lines are concurrent.

For let the first two intersect in the point $(x', y')$.
Then $A_1x' + B_1y' + C_1 = 0$ and $A_2x' + B_2y' + C_2 = 0$.
But

$$l(A_1x' + B_1y' + C_1) + m(A_2x' + B_2y' + C_2) + n(A_3x' + B_3y' + C_3) = 0,$$

since expression (32) is zero for all values of $x$ and $y$.

Hence $A_3x' + B_3y' + C_3 = 0$.

Hence $(x', y')$ lies on the third line also; that is, the three lines are concurrent.

The only use of this method lies in the fact that in some cases we can fix the values of $l$, $m$, and $n$ by inspection. The most common case is where the equations of the three straight lines are given in such forms as to require us to choose $l$, $m$, and $n$ each equal to unity.

[The student acquainted with the theory of determinants will notice that we arrive by this method at the same condition as before. For, if expression (32) is to be identically zero, we have $lA_1 + mA_2 + nA_3 = 0$, $lB_1 + mB_2 + nB_3 = 0$, $lC_1 + mC_2 + nC_3 = 0$, and the elimination of $l$, $m$, and $n$ gives the equation already found in § 21.]

*Example.—To show that the straight lines which bisect the three sides of a triangle at right angles are concurrent.*

Choose rectangular axes and let the coordinates of the vertices $A$, $B$, and $C$ be $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$.

The equation of $BC$ [§ 10 (C)] is

$$\frac{x - x_2}{x_2 - x_3} = \frac{y - y_2}{y_2 - y_3}$$

or

$$(x - x_2)(y_2 - y_3) - (y - y_2)(x_2 - x_3) = 0 \quad \ldots\ldots\ldots\ldots\ldots \; (a),$$

The coordinates of $D$, the middle point of $BC$, are $\dfrac{x_2 + x_3}{2}$, $\dfrac{y_2 + y_3}{2}$

and the general equation of any straight line through $D$ is

$$y - \frac{y_2 + y_3}{2} = m\left(x - \frac{x_2 + x_3}{2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (b).$$

To determine the value of $m$ for the straight line through $D$ perpendicular to $BC$, we interchange the coefficients of $x$ and $y$ in $(a)$ and

change the sign of one of them. Hence the equation of the line through $D$ at right angles to $BC$ is

$$(x_2-x_3)\left(x-\frac{x_2+x_3}{2}\right)+(y_2-y_3)\left(y-\frac{y_2+y_3}{2}\right)=0 \ \ldots\ldots\ c.$$

By symmetry, the equations of the lines which bisect $CA$ and $AB$ at right angles are

$$(x_3-x_1)\left(x-\frac{x_3+x_1}{2}\right)+(y_3-y_1)\left(y-\frac{y_3+y_1}{2}\right)=0 \ \ldots\ldots\ (d)$$

and

$$(x_1-x_2)\left(x-\frac{x_1+x_2}{2}\right)+(y_1-y_2)\left(y-\frac{y_1+y_2}{2}\right)=0 \ \ldots\ldots\ (e).$$

Since equations ($c$), ($d$), and ($e$) when added together vanish identically, the lines represented by these equations meet in a point.

24. **To find the condition that three points may lie on the same straight line.**

Let the coordinates of the points be $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$. Let $Ax+By+C=0$ be the equation of the straight line on which they lie. Then, since the coordinates of each point must satisfy this equation, we have

$$Ax_1+By_1+C=0 \ \ldots\ldots\ldots\ldots\ldots\ (a),$$
$$Ax_2+By_2+C=0 \ \ldots\ldots\ldots\ldots\ldots\ (b),$$
$$Ax_3+By_3+C=0 \ \ldots\ldots\ldots\ldots\ldots\ (c).$$

From ($a$) and ($b$) we have

$$\frac{A}{y_1-y_2}=\frac{B}{x_2-x_1}=\frac{C}{x_1y_2-x_2y_1}.$$

Substituting for $A$, $B$, and $C$ in (3), we have

$$x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3=0 \ \ldots\ldots\ (d)$$

as the condition that the points should be collinear. We might at once have written down equation ($d$) as the required condition, because by § 4 it expresses the condition that the area of the triangle whose vertices are the three points should be zero.

The student acquainted with the theory of determinants will recognize equation ($d$) as the expansion of the equation $\begin{vmatrix} x_1, & y_1, & 1 \\ x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \end{vmatrix}=0,$ which expresses the result of eliminating $A$, $B$, and $C$ from equations ($a$), ($b$), and ($c$).]

*Example* (i.).—*ABC is a triangle. The internal bisectors of the angles A and B cut BC and CA in D and E, and the external bisector of the angle C cuts AB in F′. Prove that the points D, E, F′ are collinear.*

Let the coordinates of $A$, $B$, $C$ be $(x_1, y_1)$, $(x_2, y_2)$, $(y_3, y_2)$, and the lengths of the sides $BC$, $CA$, $AB$ be $a$, $b$, $c$. Then, using the theorems of Euc. VI. 3 and VI. A, the coordinates of $D$, $E$, and $F'$ are (§ 3)

$$\left(\frac{bx_2+cx_3}{b+c}, \frac{by_2+cy_3}{b+c}\right), \left(\frac{cx_3+ax_1}{c+a}, \frac{cy_3+ay_1}{c+a}\right), \left(\frac{ax_1-bx_2}{a-b}, \frac{ay_1-by_2}{a-b}\right).$$

We can substitute these coordinates in equation ($d$), and the result follows; but the practised student will see by inspection of the coordinates that $F'$ divides $DE$ externally in the ratio $c+a : b+c$. (§ 3.)

*Example* (ii.).—*More generally, if A′, B′, C′ be points taken on the sides BC, CA, AB of a triangle, produced if necessary, so that*

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = -1,$$

*then the points A′, B′, C′ are collinear.*

For let $A'$ divide $BC$ in the ratio $m : n$ and let $B'$ divide $CA$ in the ratio $n : l$; then, by the conditions of the problem, $C'$ must divide $AB$ in the ratio $l : -m$, and the proof is the same as in Example i. with the substitution of $l$, $m$, $n$ for $a$, $b$, $c$.

## Exercises.

68. Find the equation of the straight line joining the point of intersection of the lines $3x+2y-5=0$ and $4x+3y+7=0$ to the point (3, 1).

69. Find the equations of the perpendiculars from the vertices on the opposite sides of a triangle, having given that the equations of its sides are $3x+y=2$, $x+2y=5$, $2x-3y+7=0$. Find also the point of intersection of these perpendiculars.

70. In the same triangle find the equations of the lines through the vertices parallel to the opposite sides.

71. Find the equations of the lines which pass through the point of intersection of the two lines $3x+4y-11=0$ and $7y-x-13=0$, and are perpendicular respectively to the lines themselves.

72. Show that the following straight lines pass through one point:

$$(b+c)x+ay=d,$$
$$(c+a)x+by=d,$$
$$(a+b)x+cy=d.$$

73. Prove that the perpendiculars from the vertices on the opposite sides of a triangle pass through one point.

**25. To find the equations of the bisectors of the angles between the straight lines whose equations referred to rectangular axes are**

$$x\cos\alpha_1+y\sin\alpha_1-p_1=0 \text{ and } x\cos\alpha_2+y\sin\alpha_2-p_2=0.$$

If $(x', y')$ is a point on either bisector, the perpendiculars from $(x', y')$ on the two lines are equal in length. Hence (§ 16) we have

$$x'\cos\alpha_1+y'\sin\alpha_1-p_1=\pm(x'\cos\alpha_2+y'\sin\alpha_2-p_2).$$

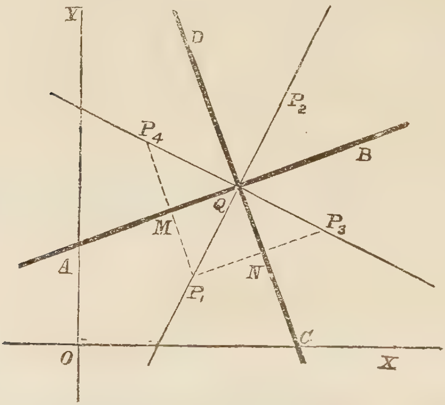



Fig. 19.

Hence every point on either bisector satisfies one or other of the equations

$$x\cos\alpha_1+y\sin\alpha_1-p_1=\pm(x\cos\alpha_2+y\sin\alpha_2-p_2)\ldots\textbf{(34)},$$

which are accordingly the equations of the bisectors, one bisector being given by the positive and the other by the negative sign.

We must now consider the question of the double sign.

If $AB$ and $CD$ be the given lines, and $P_1P_2$ and $P_3P_4$ the bisectors, one of the bisectors ($P_1P_2$ in the figure) will pass through the compartment $AQC$ in which the origin lies. Every point on the portion $QP_1$ of the bisector is on the same side of both the given lines as the origin.

Now the lengths of the perpendiculars from the origin on the two given lines (being $p_1$ and $p_2$) are the results of substituting the coordinates of the origin in the expressions

$$-(x \cos \alpha_1 + y \sin \alpha_1 - p_1)$$

and

$$-(x \cos \alpha_2 + y \sin \alpha_2 - p_2).$$

Hence the lengths of the perpendiculars from any point $(x', y')$ on $QP_1$ are the results of substituting the coordinates $x'$, $y'$ for $x$, $y$ in the same expressions. Hence the equation of $QP_1$ is

$$-(x \cos \alpha_1 + y \sin \alpha_1 - p_1) = -(x \cos \alpha_2 + y \sin \alpha_2 - p_2)$$

or

$$x \cos \alpha_1 + y \sin \alpha_1 - p_1 = x \cos \alpha_2 + y \sin \alpha_2 - p_2.$$

Thus the positive sign in (34) corresponds to the bisector ($P_1QP_2$ in the figure) which passes through the compartment in which the origin lies; and, by similar reasoning, the negative sign corresponds to the other bisector ($P_3QP_4$ in the figure).

*Oblique axes.*—In the same way the equations of the bisectors of the angles between the straight lines $x \cos \alpha_1 + y \cos(\omega - \alpha_1) - p_1 = 0$ and $x \cos \alpha_2 + y \cos(\omega - \alpha_2) - p_2 = 0$ are

$$x \cos \alpha_1 + y \cos(\omega - \alpha_1) - p_1 = \pm \{x \cos \alpha_2 + y \cos(\omega - \alpha_2) - p_2\} \quad \text{.........(35)},$$

with the same interpretation of the alternative sign as for rectangular axes.

26. **To find the equations of the bisectors of the angles between $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$.**

Reduce the equations to the perpendicular form by means of § 15, and apply the reasoning of the last article. The equations of the bisectors are found to be for rectangular axes

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm \frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}} \quad \text{.........(36)},$$

and for oblique axes

$$\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2 - 2A_1B_1 \cos \omega}} = \pm \frac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2 - 2A_2B_2 \cos \omega}} \quad \text{.........(37)}.$$

In each case the sign which makes the constant terms on both sides of like sign will correspond to the bisector which passes through the compartment containing the origin. The student should notice that the factor $\sin\omega$ has been cancelled from each side in reducing equation (37) from the expressions obtained in § 15.

*Example* (i.).—*To find the bisectors of the angles between the lines* $3x+4y=7$ *and* $8x+6y=13$, *the axes being rectangular.*

The equations of the two bisectors will be included in the formula

$$\frac{3x+4y-7}{\sqrt{3^3+4^2}} = \pm\frac{8x+6y-13}{\sqrt{8^2+6^2}},$$

or $$2(3x+4y-7) = \pm(8x+6y-13);$$

therefore the required equations are

$$2x-2y+1=0 \quad \text{and} \quad 14x+14y-27=0.$$

*Example* (ii.).—**Show that the internal bisectors of the angles of a triangle are concurrent.**

Choose the origin of coordinates *within the triangle* and let the equations of the sides $BC$, $CA$, $AB$ referred to rectangular axes be

$$x\cos\alpha_1+y\sin\alpha_1-p_1=0 \quad\ldots\ldots\ldots\ldots \quad (a),$$

$$x\cos\alpha_2+y\sin\alpha_2-p_2=0 \quad\ldots\ldots\ldots\ldots \quad (b),$$

$$x\cos\alpha_3+y\sin\alpha_3-p_3=0 \quad\ldots\ldots\ldots\ldots \quad (c).$$

The internal bisector of the angle $A$ passes through the compartment in which the origin is situated; hence its equation is

$$(x\cos\alpha_2+y\sin\alpha_2-p_2)-(x\cos\alpha_3+y\sin\alpha_3-p_3)=0 \quad\ldots\ldots \quad (d).$$

By symmetry the equations of the internal bisectors of the angles $B$ and $C$ are

$$(x\cos\alpha_3+y\sin\alpha_3-p_3)-(x\cos\alpha_1+y\sin\alpha_1-p_1)=0 \quad\ldots\ldots \quad (e),$$

and $$(x\cos\alpha_1+y\sin\alpha_1-p_1)-(x\cos\alpha_2+y\sin\alpha_2-p_2)=0 \quad\ldots\ldots \quad (f).$$

Since the total of the left-hand members of equations ($d$), ($e$), and ($f$) is identically zero, the lines represented by these equations are concurrent (§ 22).

[Observe that the proof is identical in principle with that of Euc. IV. 4.]

27. **Abridged notation.**—In Coordinate Geometry it is often possible to simplify algebraical processes by the use of abridged notation. Thus in the above example we might use the symbols $\alpha$, $\beta$, and $\gamma$ to denote the expressions

$$(x\cos\alpha_1+y\sin\alpha_1-p_1), \quad (x\cos\alpha_2+y\sin\alpha_2-p_2),$$

and $$(x\cos\alpha_3+y\sin\alpha_3-p_3).$$

With this notation the equations of the sides of the triangles would be $\alpha = 0$, $\beta = 0$, $\gamma = 0$. The equations of the internal bisectors of the angles $A$, $B$, $C$ would be

$$\beta - \gamma = 0, \quad \gamma - \alpha = 0, \quad \alpha - \beta = 0,$$

and it is evident that the total of the left-hand members of the last three equations is identically zero. Plainly the use of abridged notation cannot alter the proof; it only serves to shorten the algebraical work.

The student must remember that the simplicity of equations $(d)$, $(e)$, $(f)$ for the bisectors of the angles depends upon expressing the equations of the sides in the perpendicular form. When using abridged notation he should carefully adhere to the established convention, which is to use the letters $\alpha$, $\beta$, $\gamma$ for the abridgments of the left-hand members of equations of straight lines written in the perpendicular form—that is, for the abridgments of expressions of the form $(x \cos \alpha_1 + y \sin \alpha_1 - p_1)$ alone. For the abridgments of the left-hand members of equations of straight lines written in the general form—that is, for the abridgments of expressions of the form $(Ax + By + C)$—the letters $u$, $v$, $w$ should be used.

## Exercises.

74. Find the equations of the bisectors of the angles between the straight lines $3x - 4y + 7 = 0$ and $12x + 5y - 9 = 0$.

75. Find the bisectors of the angles formed by the following pairs of straight lines, and find also the equations of the lines joining their points of intersection to the origin :—

(i.) $y = 2x - 4$ and $y = 3x - 6$.

(ii.) $x + y - 3 = 0$ and $7x - y + 5 = 0$.

76. Find the internal bisectors of the angles of the triangle whose sides are the lines

$$4x + 3y + 7 = 0, \quad 5x + 12y + 20 = 0, \quad \text{and} \quad 3x + 4y + 8 = 0.$$

77. If $p$, $q$, $r$, the perpendiculars from a point $P$ on the sides of a given triangle, are connected by the relation $lp + mq + nr = 0$, where $l$, $m$, $n$ are constants, show that the locus of $P$ is a straight line.

[For the equation of the locus in the above notation is $l\alpha + m\beta + n\gamma = 0$, and so is of the first degree in $x$ and $y$.]

78. Prove that the external bisectors of two angles of a triangle and the internal bisector of the third are concurrent.

[For, choosing the origin within the triangle, and taking $\alpha = 0$, $\beta = 0$, $\gamma = 0$ for the abridged equations of the sides expressed in perpendicular form, the equations of the external bisectors of the angles $A$ and $B$ and the internal bisector of the angle $C$ are $\beta + \gamma = 0$, $\gamma + \alpha = 0$, $\alpha - \beta = 0$. The theorem follows from Art. 13, by choosing $l = 1$, $m = -1$, $n = 1$.]

28. **The homogeneous quadratic equation of the second degree in $x$ and $y$ represents two straight lines through the origin.**

Let the equation be

$$ax^2+2hxy+by^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a).$$

Solving as a quadratic in $y$, we have

$$y=\frac{-h+\sqrt{(h^2-ab)}}{b}x \quad \text{and } y=\frac{-h-\sqrt{(h^2-ab)}}{b}x,$$

which are the equations of two straight lines through the origin. These two straight lines form the locus represented by equation $(a)$; for the coordinates of any point on either straight line will satisfy equation $(a)$.

These lines are real if $h^2>ab$, coincident if $h^2=ab$, imaginary if $h^2<ab$.

The above results hold equally for rectangular and oblique axes.

Note carefully that $h$ is the coefficient of $2xy$ and not of $xy$.

29. **To find the angle between the two straight lines represented by the equation**

$$ax^2+2hxy+by^2=0.$$

If the equations of the straight lines are $y-m_1x=0$ and $y-m_2x=0$, the expression $ax^2+2hxy+by^2$, when divided by $b$ so as to reduce the coefficient of $y^2$ to unity, must be the product of the factors $y-m_1x$ and $y-m_2x$;

$$\therefore \quad m_1+m_2=-2h/b, \quad m_1m_2=a/b.$$

*Rectangular axes.*—If $\phi$ be the angle between the lines,

$$\tan\phi=(m_1-m_2)/(1+m_1m_2). \qquad (\S\ 19)$$

Now $(m_1-m_2)^2=(m_1+m_2)^2-4m_1m_2=4(h^2-ab)/b^2$;

$$\therefore \quad \tan\phi=\pm\frac{2\sqrt{h^2-ab}}{a+b} \quad \ldots\ldots\ldots\ldots\ldots (38).$$

The double sign corresponds to the fact that the angle between the lines may be equally well taken as being $\phi$ or its supplement.

The **condition for coincidence of the lines** is

$$h^2=ab \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39);$$

that is, that the expression $ax^2+2hxy+by^2$ should be a perfect square.

The **condition for perpendicularity** is

$$\boldsymbol{a+b=0} \qquad \ldots\ldots\ldots\ldots\ldots \textbf{(40)}.$$

*Oblique axes.*—Here

$$\tan\phi = \frac{(m_1-m_2)\sin\omega}{1+(m_1+m_2)\cos\omega+m_1m_2},$$

whence
$$\boldsymbol{\tan\phi = \pm\frac{2\sqrt{(h^2-ab)}\sin\omega}{a+b-2h\cos\omega}} \qquad \ldots\ldots\ldots \textbf{(41)}.$$

The condition for coincidence is $\boldsymbol{h^2=ab}$, as before. The condition for perpendicularity is

$$\boldsymbol{a+b-2h\cos\omega=0} \qquad \ldots\ldots\ldots\ldots\ldots \textbf{(42)}.$$

The conditions for coincidence and perpendicularity should be remembered.

**30. To find the equation of the bisectors of the angles between the pair of lines $ax^2+2hxy+by^2=0$, the axes being rectangular.**

Suppose the lines are

$$y-m_1x=0, \quad y-m_2x=0.$$

Since from any point on a bisector the perpendiculars on the lines are equal, we have

$$\frac{y-m_1x}{\sqrt{1+m_1^2}} = \pm\frac{y-m_2x}{\sqrt{1+m_2^2}}$$

where the upper sign refers to one bisector, the lower one to the other. Thus the equation

$$\frac{(y-m_1x)^2}{1+m_1^2} = \frac{(y-m_2x)^2}{1+m_2^2}$$

represents both bisectors. Now it may be written

$$(1+m_2^2)(y-m_1x)^2-(1+m_1^2)(y-m_2x)^2=0,$$

or $x^2\{m_1^2(1+m_2^2)-m_2^2(1+m_1^2)\}$

$$-2xy\{m_1(1+m_2^2)-m_2(1+m_1^2)\}+y^2(1+m_2^2-1-m_1^2)=0$$

or

$$x^2(m_1^2-m_2^2)-2xy\{(m_1-m_2)(1-m_1m_2)\}+y^2(m_2^2-m_1^2)=0$$

$$\therefore \quad (x^2-y^2)(m_1+m_2)=2xy(1-m_1m_2), \quad \text{since } m_1-m_2\neq 0.$$

But $$m_1+m_2 = -2h/b, \quad m_1m_2 = a/b,$$
and therefore the equation becomes
$$(x^2-y^2)(-2h/b) = 2xy(1-a/b)$$
or
$$(x^2-y^2)h = xy(a-b);$$
*i.e.*,
$$\frac{x^2-y^2}{a-b} = \frac{xy}{h} \quad \text{..................... } \mathbf{(43)}.$$

In this form the equation, which is extremely important, can be easily remembered.

*Example.—To find the equation of the lines bisecting the angles between the pair of lines* $3x^2+xy-2y^2 = 0$.

Using the formula $$\frac{x^2-y^2}{a-b} = \frac{xy}{h},$$

we get the equation $$\frac{x^2-y^2}{3-(-2)} = \frac{xy}{\frac{1}{2}}$$

or $$x^2-10xy-y^2 = 0.$$

**Exercises.**

79. Find the separate equations of the straight lines whose joint equation is
$$x^2-5xy+6y^2 = 0.$$

80. State what loci are represented by

(*a*) $xy = 0$. (*b*) $y^2 = 0$. (*c*) $x^2+y^2 = 0$.
(*d*) $xy+y^2 = 0$. (*e*) $x^2-2xy+y^2 = 0$. (*f*) $y^2+1 = 0$.
(*g*) $(x-a)^2+(y-b)^2 = 0$. (*h*) $xy-3x-2y+6 = 0$.

81. Find the angle between the lines whose joint equation is
$$2x^2-3xy+y^2 = 0,$$
the axes being rectangular.

82. Find the straight lines represented by the equation
$$y^2-xy-6x^2 = 0,$$
and find the angle between them.

83. Find the angle between the lines represented by the equation
$$39x^2-96xy+11y^2 = 0.$$

84. Show that the pair of lines $x^2+2xy\sec\alpha+y^2 = 0$ are always real, and that the angle between them is $\alpha$.

85. Show from their equation that the bisectors of the angles between $ax^2+2hxy+by^2 = 0$ are at right angles.

86. If $a = b$, show that the lines $ax^2+2hxy+by^2 = 0$ have the same bisectors of angles as the axes, and hence that one of the lines makes the same angle with $OX$ as the other does with $OY$.

**31. Change of axes.—To change the origin of coordinates without changing the directions of the axes.**

Let $OX$, $OY$ be the old axes. Let $O'X'$, $O'Y'$ be the new axes parallel to $OX$, $OY$. Draw $OH$ and $HO'$, the abscissa and ordinate of $O'$ referred to the old axes and let the coordinates of $O'$ referred to the old axes, be $h$, $k$, so that

$$OH = h, \quad HO' = k.$$

Let $P$ be any point, $OM$ and $MM'P$ its abscissa and ordinate referred to the old axes, $O'M'$ and $M'P$ its abscissa and ordinate referred to the new axes.

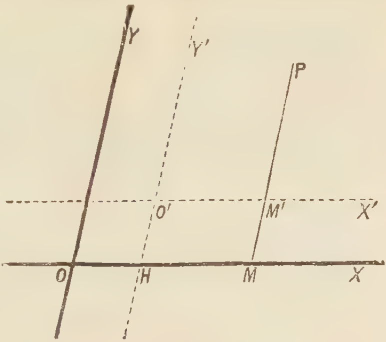



Fig. 20.

Let $x$, $y$ be the coordinates of $P$ referred to the old axes $x'$, $y'$; its coordinates referred to the new axes. We wish to transform an equation in $x$ and $y$ into the corresponding equation in $x'$ and $y'$. Hence we want to express the old coordinates $x$, $y$ in terms of the new ones $x'$, $y'$, so that the transformation can be performed by direct substitution. We have clearly

$$OM = h + O'M', \quad MP = k + M'P$$

or

$$x = x' + h, \qquad y = y' + k \;\ldots\ldots\ldots\ldots (44).$$

We have therefore to write $(x'+h)$ for $x$ and $(y'+k)$ for $y$ in the equation which we wish to transform. We thus get an equation in $x'$, $y'$. As $x'$, $y'$ are our new current coordinates, we get our transformed equation by suppressing the accents in the equation in $x'$, $y'$. Clearly therefore we can perform the transformation in one step by writing $(x+h)$ for $x$ and $(y+k)$ for $y$ in the equation which we wish to transform.

The proof applies equally to rectangular and oblique axes.

**32. To find the condition that the general equation of the second degree may represent two straight lines.**

Let the equation be

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \ldots \quad \mathbf{(45)}.$$

[The student should notice this equation. It is in reality symmetrical, the left-hand member being the form taken by the symmetrical expression

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

when $z$ is equal to unity. Consequently, in equation $(a)$, $f$ corresponds to $y$, $g$ to $x$, and $h$ to $xy$. Note that $g$ is the coefficients of $2x$, and not of $x$, and similarly with regard to $f$ and $h$.]

If equation (45) represents two straight lines, let them intersect in the point $(x_1, y_1)$.

Transform the equation by referring it to parallel axes through $(x_1, y_1)$; it then takes the form (§ 31)

$$a(x+x_1)^2+2h(x+x_1)(y+y_1)+b(y+y_1)^2+2g(x+x_1)$$
$$+2f(y+y_1)+c=0 \quad \ldots\ldots \quad (a).$$

Referred to the new axes the equation is the equation of two straight lines through the origin, and must therefore be a homogeneous quadratic expression in $x$ and $y$. Hence the coefficient of $x$, the coefficient of $y$, and the constant term must each vanish in equation $(a)$.

Hence

$$ax_1+hy_1+g=0 \quad \ldots\ldots \quad (b),$$
$$hx_1+by_1+f=0 \quad \ldots\ldots \quad (c),$$
$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \quad \ldots\ldots \quad (d).$$

Equation $(d)$ may be written

$$x_1(ax_1+hy_1+g)+y_1(hx_1+by_1+f)+gx_1+fy_1+c=0;$$
$$\therefore \quad gx_1+fy_1+c=0 \quad \ldots\ldots \quad (e).$$

Eliminating $x_1$ and $y_1$ from equations $(b)$, $(c)$, and $(e)$, we have

$$abc+2fgh-af^2-bg^2-ch^2=0 \quad \ldots\ldots \quad \mathbf{(46)},$$

which is the condition required.

Equation (46), being the result of eliminating $x_1$ and $y_1$ from the equations

$$ax_1 + by_1 + g = 0, \quad hx_1 + by_1 + f = 0, \quad gx_1 + fy_1 + c = 0,$$

is, of course, the expansion of the equation $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0.$

The equation of the two straight lines referred to the new axes is $ax^2 + 2hxy + by^2 = 0$. The straight lines are therefore parallel to those whose equations referred to the old axes are $ax^2 + 2hxy + by^2 = 0$, and the conditions of parallelism and perpendicularity are given by the results of § 29, and depend only on the terms of the second degree.

The result of the preceding paragraph may be verified as follows:—Equation (45), if it represents two straight lines, must be the product of two linear factors in $x$ and $y$. Hence in solving for $y$ in terms of $x$ the roots must be rational functions of $x$. Now the equation, rearranged in descending powers of $y$ is

$$by^2 + 2(hx + f)y + ax^2 + 2gx + c = 0.$$

Hence

$$y = \frac{-(hx+f) \pm \sqrt{(hx+f)^2 - b(ax^2 + 2gx + c)}}{b}.$$

The quantity under the radical must be a perfect square; *i.e.* $(h^2 - ab)x^2 + 2(hf - bg)x + (f^2 - bc)$ must be a perfect square; hence $(h^2 - ab)(f^2 - bc) = (hf - bg)^2$, which reduces to

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

### Exercises.

87. Find what the following equations become when the origin is transformed to the point (1, 1).

(*a*) $x^2 + xy - 3x - y + 2 = 0$. (*b*) $xy - y^2 - x + y = 0$.

(*c*) $xy - x - y + 1 = 0$. (*d*) $x^2 - y^2 - 2x + 2y = 0$.

88. Transform the equation $\frac{x}{a} + \frac{y}{b} - 1 = 0$ by referring it to parallel axes through the point $(a, 0)$.

89. Transform the equation $x\cos\alpha + y\sin\alpha = p$ by referring it to parallel axes through the point $(p\cos\alpha, p\sin\alpha)$.

90. Transform the origin to the point $(-g, -f)$ in the equation
$$x^2+y^2+2gx+2fy+c=0.$$

91. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines, show that they intersect in the point
$$\left(\frac{hf-bg}{ab-h^2}, \frac{hg-af}{ab-h^2}\right).$$
[Solve for $x_1$ and $y_1$ from equations $(b)$ and $(c)$.]

92. Perform the corresponding transformation when the equation does not represent a pair of straight lines.

[Here we can still, by choosing the coordinates of the new origin to satisfy equations $(b)$ and $(c)$, get a transformed equation with zero coefficients for $x$ and $y$, but the constant term, instead of vanishing, will be $gx_1+fy_1+c$ where $x_1=\frac{hf-bg}{ab-h^2}$, $y_1=\frac{hg-af}{ab-h^2}$; and the transformed equation will be
$$ax^2+2hxy+by^2+\frac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}=0.]$$

Note this result hereafter as an introduction to Chapter VI.

[The student acquainted with the theory of determinants will note in Ex. 2 that the coordinates of the new origin are $\frac{G}{C}, \frac{F}{C}$, and that the transformed equation is $ax^2+2hxy+by^2+\frac{\Delta}{C}=0$ where $\Delta\equiv\begin{vmatrix} a, h, g \\ h, b, f \\ g, f, c\end{vmatrix}$ and capitals denote the minors of the corresponding small letters.]

93. Show that the equation $x^2-y^2-x+3y-2=0$ represents a pair of straight lines; find them, and show that they are at right angles.

94. Draw the locus of the equation $(x+y-1)^2-4x^2=0$, and find the point of intersection of the lines which it represents.

95. Show that the equation $x^2-5xy+4y^2+x+2y-2=0$ represents a pair of straight lines, and find their point of intersection and their equations.

96. By transferring the origin to the point $(-1, 2)$ show that the equation $x^2+xy-y^2+5y-5=0$ represents a pair of straight lines. Then, by changing the origin back again to its original position, show that the equation of the bisectors of the angles between them is
$$x^2-4xy-y^2+10x+5=0.$$

97. To what point must the origin be transferred so as to remove the terms of the first degree in $x^2+xy+2y^2-7x-5y+12=0$, and what does the equation then become?

**33. To change from one set of rectangular axes to another, the origin being unaltered.**

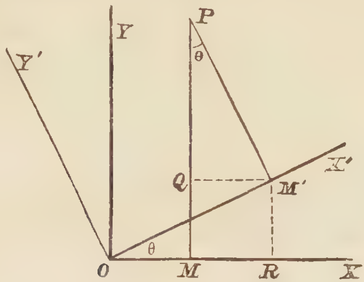



Fig. 21.

Let the new axes $OX'$, $OY'$ be inclined to the old axes $OX$, $OY$ at an angle $\theta$.

Let the coordinates of any point $P$ referred to the old axes be $(x, y)$ and referred to the new axes $(x', y')$. As explained in § 31, what we want to do is to express $x$ and $y$ in terms of $x'$ and $y'$.

Draw $PM$, $PM'$ perpendicular to $OX$, $OX'$; draw $MR$ perpendicular to $OX$ and $M'Q$ perpendicular to $PM$.

Then $$OM = OR - QM'$$

and $$PM = RM' + QP.$$

$$\therefore \left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots \quad (47).$$

Hence, on suppressing accents, the transformed equation will be obtained by substituting $(x\cos\theta - y\sin\theta)$ for $x$ and $(x\sin\theta + y\cos\theta)$ for $y$ in the original equation.

*Example* (i.)—*The axes being rectangular, show that the equations connecting the old coordinates $x$, $y$, with new coordinates $x'$, $y'$, referred to rectangular axes inclined at an angle $\theta$ to the old axes with the new origin at the point $(h, k)$ are*

$$x = h + x'\cos\theta - y'\sin\theta,$$

$$y = k + x'\sin\theta + y'\cos\theta.$$

[This follows by first transferring to parallel axes through $(h, k)$, and then turning the axes round · or directly from a figure.]

*Example* (ii.)—*Find what the equation $x^2+4xy+y^2=0$ becomes when the axes, supposed rectangular, are turned through an angle* 45°.

Here $$\alpha = 45;$$

$$\therefore \quad x = \frac{x'-y'}{\sqrt{2}}, \quad y = \frac{x'+y'}{\sqrt{2}}.$$

Hence $$\frac{(x'-y')^2}{2} + \frac{4(x'-y')(x'+y')}{2} + \frac{(x'+y')^2}{2} = 0,$$

or $$(x'-y')^2 + 4(x'^2-y'^2) + (x'+y')^2 = 0;$$

*i.e.* $$6x'^2 - 2y'^2 = 0 \quad \text{or} \quad y' = \pm x'\sqrt{3}.$$

This equation shows us that the two lines represented are inclined at angles of 60° on each side of the line $OX'$. Thus, if $OA$, $OB$ be the lines, we have

$$AOX' = BOX' = 60°;$$

$$\therefore \quad AOX = 15°, \quad BOY = 15°,$$

giving the positions of the lines referred to the old axes.

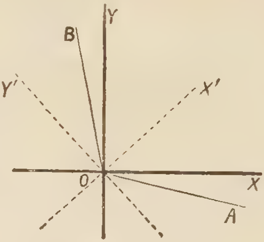


Fig. 22.

The reader should verify, by solving for $y$ in terms of $x$, that the equation

$$x^2+4xy+y^2=0$$

really does represent two lines one of which makes an angle of 15° with $OX$, and the other the same angle with $OY$.

## Exercises.

98. Find what the following equations become when the point $(-1, 1)$ is taken for the new origin :—

(i.) $x=0$; (ii.) $y=0$; (iii.) $lx+my+1=0$; (iv.) $x^2-y^2=0$.
(v.) $3x^2-4xy+y^2=0$; (vi.) $2x^2+3xy+4y^2-x+y+1=0$.

99. Write down the formulæ necessary for turning the axes round through (i.) 30°, (ii.) 225°.

100. If the axes are turned through an angle $\tan^{-1}\frac{1}{2}$, express the old coordinates in terms of the new; also the new in terms of the old.

101. Find what the equation $11x^2+16xy-y^2=0$ becomes on turning the axes round through an angle $\tan^{-1}\frac{1}{2}$.

102. What are the formulæ of transformation for turning the axes round through a right angle? Verify them from first principles.

34. We have seen that it is usually simplest, in effecting transformations consequent on change of axes, to express the old coordinates $(x, y)$ in terms of the new $(x', y')$. There is one case, however, in which it is

simpler to express $x'$ and $y'$ by inspection in terms of $x$ and $y$. Suppose we are given an equation, referred to rectangular axes, in the form

$$a^2(x\cos\alpha_1+y\sin\alpha_1-p_1)^2+b^2(x\cos\alpha_2+y\sin\alpha_2-p_2)^2=a^2b^2$$

where $\alpha_2-\alpha_1=\frac{1}{2}\pi$, and that we wish to refer the equation to the straight lines whose equations are

$$x\cos\alpha_1+y\sin\alpha_1-p_1=0 \quad\text{and}\quad x\cos\alpha_2+y\sin\alpha_2-p_2=0$$

as new axes of $x$ and $y$ respectively. (Note that these new axes will not be rectangular unless $\alpha_2-\alpha_1=\frac{1}{2}\pi$.

We have $x'$ = perpendicular from $(x, y)$ on new axis of $y$

$$= \pm(x\cos\alpha_2+y\sin\alpha_2-p_2).$$

Similarly $\quad y'=\pm(x\cos\alpha_1+y\sin\alpha_1-p_1).$

Hence the equation now becomes

$$a^2y'^2+b^2x'^2=a^2b^2;$$

or, suppressing accents and reducing

$$x^2/a^2+y^2/b^2=1.$$

The ambiguous sign above merely means that we are at liberty to choose whichever side we please for the positive side of either of the new axes. Frequently, however, we shall find that in the transformed equation $x$ and $y$ will only occur in even powers, in which case the curve will be symmetrical with respect to both of the new axes, and the question of the ambiguous sign will be immaterial.

In performing this transformation the student must carefully remember to see that the proposed new axes are at right angles, and that their equations are reduced to the perpendicular form.

*Example.—Transform the equation*

$$(3x-4y-10)^2+4(4x+3y+15)^2=25 \;\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

*(referred to rectangular axes) in this manner.*

The straight lines $3x-4y-10=0$, $4x+3y+15=0$ are evidently rectangular, since in passing from the first to the second we have to interchange the coefficients of $x$ and $y$, and change the sign of one of them. Take the former line for the new axis of $x$ and the latter for the new axis of $y$. To reduce their equations to the perpendicular form, we must divide the expression inside each bracket by $\sqrt{3^2+4^2}$ or 5. Equation (A) now becomes

$$(\tfrac{3}{5}x-\tfrac{4}{5}y-2)^2+(\tfrac{4}{5}x+\tfrac{3}{5}y+3)^2=1.$$

Transforming to the new axes, this becomes $y'^2+4x'^2=1$.

The new equation is therefore $4x^2+y^2=1$.

**Exercise.**

103. Transform the equation

$$(5x+12y-39)^2 = 52(12x-5y+52)$$

(referred to rectangular axes), taking the lines $5x+12y-39=0$ and $12x-5y+52=0$ as the new rectangular axes of $x$ and $y$ respectively, the side of the new axis of $y$ on which the old origin is situated being taken as the positive side.

35. **To effect any change of axes we replace the old coordinates by linear functions of the new ones.**

I. Let the old axes be rectangular.

Let $P$ be any point; $OX$, $OY$ the old axes and $O'X'$, $O'Y'$ the new axes.

Let the equations of the new axes referred to the old be

$$l_1x+m_1y+n_1=0, \quad l_2x+m_2y+n_2=0,$$

and let the angle between the new axes be $\omega$.

Then $y'\sin\omega$ = perpendicular from $P$ on $O'X'$,

$$= \pm\frac{l_1x+m_1y+n_1}{\sqrt{l_1^2+m_1^2}},$$

Similarly $$x'\sin\omega = \pm\frac{l_2x+m_2y+n_2}{\sqrt{l_2^2+m_2^2}}.$$

(We have already explained the meaning of the ambiguous sign.)

Hence $x'$, $y'$ are linear functions of $x$ and $y$, say

$$x'=px+qy+r, \quad y'=p'x+q'y+r'.$$

Solving for $x$ and $y$, we see that $x$ and $y$ are linear functions of $x'$ and $y'$.

II. Let the old axes be oblique and inclined to each other at an angle $\Omega$.

The only variation from the preceding proof consists in substituting the expressions $l_1^2+m_1^2-2l_1m_1\cos\Omega$ and $l_2^2+m_2^2-2l_2m_2\cos\Omega$ for $l_1^2+m_1^2$ and $l_2^2+m_2^2$ under the radicals in the denominators of the expressions for $y'\sin\omega$ and $x'\sin\omega$.

36. **The degree of an equation cannot be altered by any change of axes.**

I. *The degree cannot be increased;* for suppose $ax^ly^m$ the term of highest degree. It will be replaced by an expression of the form $a(p_1x'+q_1y'+r_1)^l.(p_2x'+q_2y'+r_2)^m$, and the degree of the term of highest degree in $x'$ and $y'$ in this expression is $l+m$, just as in $ax^ly^m$.

II. *The degree cannot be diminished;* for, since on transferring back again we must get back to the original equation, the degree would be increased by this transformation. But, since $x'$, $y'$ are linear functions of $x$ and $y$, this is impossible by I.

**37. If by any change of axes, the origin remaining unaltered, the expression**

$$ax^2+2hxy+by^2 \quad \textbf{becomes} \quad a'x^2+2h'xy+b'y^2,$$

**then**
$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'}$$

**and**
$$\frac{ab-h^2}{\sin^2\omega}=\frac{a'b'-h'^2}{\sin^2\omega'}$$

**where $\omega$, $\omega'$ are the angles between the axes in the two cases.**

For the expression $x^2+y^2+2xy\cos\omega$ denotes the square of the distance of the point $(x, y)$ from the origin. This distance remains unaltered, since the origin remains unaltered; hence the expression $x^2+y^2+2xy\cos\omega$ transforms into $x^2+y^2+2xy\cos\omega'$. Hence, by hypothesis,

$$ax^2+2hxy+by^2+\lambda(x^2+y^2+2xy\cos\omega) \;\ldots\ldots\; (A)$$

transforms into

$$a'x^2+2h'xy+b'y^2+\lambda(x^2+y^2+2xy\cos\omega') \;\ldots\ldots\; (B).$$

Now, if expression (A) is a perfect square—that is to say, if it is of the form $(px+qy)^2$—expression (B) must be a perfect square also; for the transformation consists in substituting for $(px+qy)$ an expression of the first degree in $x$ and $y$ (§ 35). The conditions that expressions (A) and (B) should be perfect squares, namely,

$$(a+\lambda)(b+\lambda)-(h+\lambda\cos\omega)^2=0,$$
$$(a'+\lambda)(b'+\lambda)-(h'+\lambda\cos\omega')^2=0.$$

must therefore give the same value for $\lambda$ on solution. Hence, comparing the coefficients of $\lambda^2$ and $\lambda$ and the constant terms, we have

$$\frac{\sin^2\omega}{\sin^2\omega'}=\frac{a+b-2h\cos\omega}{a'+b'-2h'\cos\omega'}=\frac{ab-h^2}{a'b'-h'^2},$$

that is
$$\left.\begin{aligned}\frac{a+b-2h\cos\omega}{\sin^2\omega}&=\frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'}\\ \text{and}\qquad \frac{ab-h^2}{\sin^2\omega}&=\frac{a'b'-h'^2}{\sin^2\omega'}\end{aligned}\right\}\ldots(\mathbf{47}).$$

Cor. If both sets of angles are rectangular, we have

$$a+b=a'+b' \quad\text{and}\quad ab-h^2=a'b'-h'^2\ \ldots(\mathbf{48}).$$

**Invariants.**—The preceding theorem may be stated in a different way. The result shows that the expressions

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}\quad\text{and}\quad\frac{ab-h^2}{\sin^2\omega}$$

are the same as the corresponding expressions after the axes are changed, that is, their values are unaltered by any change of axes. They are therefore called **invariants.**

38. **To find the equation of the straight lines joining the origin to the points of contact of the curve $ax^2+2hxy+by^2+2gx+2fy+c=0$ with the straight line $lx+my+n=0$.**

Rule.—Make the first equation homogeneous in $x$ and $y$ by the aid of the second equation written in the form

$$-(lx+my)/n=1,$$

and we get

$$ax^2+2hxy+by^2-2(gx+fy)\left(\frac{lx+my}{n}\right)+c\left(\frac{lx+my}{n}\right)^2=0 \qquad\ldots\ldots(a)$$

as the required equation. For this equation is a homogeneous one of the second degree in $x$ and $y$, and therefore represents two straight lines through the origin. Further, it is satisfied by the coordinates of the points whose coordinates satisfy the given equations, that is, of the points where the straight line cuts the curve. Hence equation $(a)$ is the equation required.

*Example.—To find the equation of the straight lines joining the origin to the point where the straight line $x+y+2=0$ meets the curve*

$$x^2+xy+y^2+x+3y+1=0.$$

Here $$-\frac{x+y}{2}=1;$$

and hence the homogeneous equation is

$$x^2+xy+y^2+(x+3y)\left(-\frac{x+y}{2}\right)+\left(\frac{x+y}{2}\right)^2=0,$$

or $$4(x^2+xy+y^2)-2(x+3y)(x+y)+(x+y)^2=0;$$

*i.e.* $$3x^2-2xy-y^2=0 \quad \text{or} \quad (x-y)(3x+y)=0.$$

Thus the two lines are $x-y=0$, $3x+y=0$.

## Exercises.

104. Find the equation of the lines joining the origin to the points in which the line $x+2y=2$ meets the curve $x^2+xy+2y^2=3$.

105. Find the equation of the lines joining the origin to the points of intersection of the straight line $6x+8y-75=0$ and the circle

$$x^2+y^2+6x+8y=150.$$

106. Find the equation of the lines joining the origin to the points of intersection of the curve $x^2+y^2-4x-5=0$ and the straight line $y=3$. Explain the result.

## Illustrative Examples.

(i) *$A$ and $B$ are two fixed points. Find the locus of a point $P$ which moves so that $PA^2+PB^2$ is constant.*

Let $PA^2+PB^2=c^2$. Take rectangular axes with $AB$ for axis of $x$, and for symmetry take its middle point $O$ for origin. Let the length of $AB$ be $2a$. Then the coordinates of $A$ are $(-a, 0)$ and of $B$ $(a, 0)$. Let $(x, y)$ be the coordinates of $P$.

Then $$PA^2=(x+a)^2+y^2, \quad PB^2=(x-a)^2+y^2;$$

$$\therefore \quad PA^2+PB^2=2(x^2+y^2+a^2);$$

$$\therefore \quad x^2+y^2=\tfrac{1}{2}c^2-a^2.$$

That is, $OP^2=\frac{1}{2}c^2-a^2$, or the locus of $P$ is a circle with its centre at the middle point of $AB$.

(ii.) **Find the equation of the two lines through the origin perpendicular to the lines $ax^2+2hxy+by^2=0$, the axes being rectangular.**

Suppose the given lines are $y-m_1x=0$ and $y-m_2x=0$; so that

$$m_1+m_2=-\frac{2h}{b}, \quad m_1m_2=\frac{a}{b};$$

the lines perpendicular to these are

$$m_1y + x = 0 \quad \text{and} \quad m_2y + x = 0 \qquad (\S\ 19)$$

respectively, giving the equation

$$(m_1y + x)(m_2y + x) = 0\ ;$$

*i.e.* $$m_1m_2y^2 + (m_1 + m_2)xy + x^2 = 0.$$

Finally, putting in the values of $m_1 + m_2$ and $m_1m_2$, we find, as the required equation, $\quad bx^2 - 2hxy + ay^2 = 0.$

(iii.) *$AA'$, $BB'$ are two given finite straight lines. Find the locus of a point $P$ which moves so that the sum of the areas $APA'$, $BPB'$ is constant.*

Let this sum be $\Delta$. Let $AA'$, $BB'$ intersect in $O$. Take $OAA'$ and $OBB'$ for axes of $x$ and $y$. These will in general be oblique. Let the angle between them be $\omega$. Let $OA = a$, $OA' = a'$, $OB = b$, $OB' = b'$. Let $a'$ be greater than $a$ and $b'$ greater than $b$. Let $x$, $y$ be the coordinates of $P$, assumed to lie in the positive quadrant. The coordinates of the vertices $P$, $A$, $A'$ of the triangle $PAA'$, which, upon the above assumptions, will occur in the counterclockwise order, are $(x, y)$, $(a, 0)$, $(a', 0)$.

Hence $\quad$ the area $PAA' = \frac{1}{2}y(a' - a)\sin\omega. \qquad (\S\ 4)$

Similarly $\quad$ the area $PB'B = \frac{1}{2}x(b' - b)\sin\omega\ ;$

$$\therefore \quad \tfrac{1}{2}y(a' - a)\sin\omega + \tfrac{1}{2}x(b' - b)\sin\omega = \Delta,$$

or $$\frac{x}{a' - a} + \frac{y}{b' - b} = \frac{2\Delta}{(a' - a)(b' - b)\sin\omega} \;\ldots\ldots\ldots\ldots\ldots\; (1),$$

the equation of a straight line parallel to

$$\frac{x}{a' - a} + \frac{y}{b' - b} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

NOTE.—When $P$ crosses $AA'$, $y$ becomes negative, and, in the analytical work, the area $PAA'$ becomes negative, the vertices $P$, $A$, $A'$ now occurring in the clockwise order. To interpret this part of the locus geometrically we must express the condition of the problem as follows :—The *difference* of the areas $PAA'$ and $PBB'$, regarded as having magnitude only, and not sign, is equal in magnitude to $\Delta$.

(iv.) *A flagstaff $HK$ on the top of a perpendicular cliff $OH$ subtends an angle $\alpha$ at each of two points $A$ and $B$ distant $a$ and $b$ horizontally in the same straight line from the base $O$ of the cliff. Prove that the height of the flagstaff is $(a + b)\tan\alpha$.*

Take $OAB$ for axis of $x$, $OHK$ for axis of $y$. These axes are rectangular.

Then $OA = a$, $OB = b$. Let $OH = c$, $HK = h$.

Then the coordinates of $H$, $K$ are $(0, c)$ and $(0, c + b)$.

The equation of the line joining $H$ to a point $(x_1, 0)$ on $OAB$ is

$$\frac{x}{x_1} + \frac{y}{c} = 1. \qquad (\S\ 10\ \text{D})$$

The equation of the line joining $K$ to the same point is

$$\frac{x}{x_1} + \frac{y}{c+h} = 1.$$

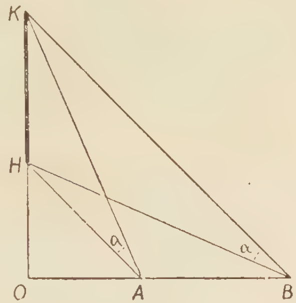


Fig. 23.

The angle $\phi$ between these two lines is determined by the equation

$$\tan\phi = \frac{-\frac{c}{x_1} + \frac{c+h}{x_1}}{1 + \frac{c(c+h)}{x_1^2}}$$

$$= \frac{hx_1}{x_1^2 + ch + c^2} \ (\S\ 19)$$

or $x_1^2 - x_1 h \cot\phi + ch + c^2 + 0$ ... (1).

By the conditions of the problem when $\phi = \alpha$, the two roots of this quadratic in $x_1$ are $a$ and $b$;

$$\therefore \quad a + b = \text{sum of roots when } \phi = \alpha$$
$$= h \cot \alpha,$$

which gives the result required.

[Note that it is not necessary to solve the quadratic for the two separate values of $x_1$.]

(v.) *$ABCD$ is a quadrilateral. $AB$ and $CD$ meet in $O$, $AC$ and $BD$ in $P$ and $AD$ and $BC$ in $Q$. $PQ$ cuts $AB$ in $M$ and $CD$ in $N$. Show that*

$$\frac{1}{OA} + \frac{1}{OB} = \frac{2}{OM} \quad \textit{and} \quad \frac{1}{OC} + \frac{1}{OD} = \frac{2}{ON}.$$

Let $OA = a$, $OB = b$, $OC = c$, $OD = d$.
Take $OAB$ and $OCD$ for oblique axes of $x$ and $y$.

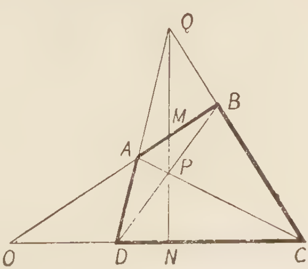


Fig. 24.

The equation of $AD$ is

$$x/a + y/d - 1 = 0 \ \ldots \ (a),$$

and of $BC$

$$x/b + y/c - 1 = 0 \ \ldots \ (b),$$

and of $AC$

$$x/a + y/c - 1 = 0 \ \ldots \ (c),$$

and of $BD$

$$x/b + y/d - 1 = 0 \ \ldots \ (d).$$
(§ 10 D)

The equation of $PQ$, which is a line through the intersection of $AD$ and $BC$, must be of the form

$$(x/a + y/d - 1) + k(x/b + y/c - 1) = 0 \ldots\ldots\ldots\ldots \ (e).$$

Also, since $PQ$ passes through the intersection of $AC$ and $BD$, its equation must be of the form

$$(x/a+y/c-1)+k'(x/b+y/d-1)=0 \quad \ldots\ldots\ldots\ldots\ (f).$$

To obtain the equation of $PQ$ we have to choose $k$ and $k'$, so that equations $(e)$ and $(f)$ may represent the same straight line; clearly this necessitates $k=1$, $k'=1$; the equation of $PQ$ is therefore

$$x(1/a+1/b)+y(1/c+1/d)-2=0.$$

Where $PQ$ cuts $AB$, $x=OM$, $y=0$; therefore $2/OM=1/a+1/b$, and similarly $2/ON=1/c+1/d$.

## MISCELLANEOUS EXERCISES ON CHAP. I.

107. Find the coordinates of the points dividing the line joining $(1, 1)$ and $(-3, 4)$ internally in the ratio $2:1$ and externally in the ratio $1:3$.

108. Show that, whether the axes are rectangular or oblique, the line joining $(x, y)$ to $(-x, y)$ is bisected by the axis of $y$.

109. Trace the loci (i.) $\theta=\frac{\pi}{4}$, (ii.) $r=3$, (iii.) $\sin 2\theta=0$.

110. Find the locus of a point such that the square of its distance from the point $(4, 0)$ is four times the square of its distance from the point $(1, 0)$.

111. Find the locus of a point the squares of whose distances from the origin and the point $(4, 3)$ are equal.

112. Trace the locus of the equation $4x^2=9y^2$.

113. Trace the locus of the equation $x^2+y^2=36$.

114. If the axes are inclined at an angle $\omega$ and $P$ be the point $(x_1, y_1)$, show that, if $PM$, $PN$ be perpendicular to the axes, then

$$OM=x_1+y_1\cos\omega;\ ON=x_1\cos\omega+y_1.$$

Deduce the equation of $MN$.

115. If $MN$ in the last question passes through a fixed point $(p, q)$, find the equation of the locus of $P$.

116. The equation of a straight line such that the portion of it intercepted between the axes is bisected at the point $(x_1, y_1)$ is

$$\frac{x}{2x_1}+\frac{y}{2y_1}=1.$$

117. A straight line passes through a fixed point $(x_0, y_0)$. Show that the equation of the locus of the middle point of it intercepted between the axes is $\qquad \dfrac{x_0}{2x}+\dfrac{y_0}{2y}=1.$

118. Find the tangent of the angles between the following pairs of lines, the axes being rectangular : —

(i.) $x^2 - y^2 = 0$; (ii.) $2x^2 + 3xy - 4y^2 = 0$;

(iii.) $x^2 + 4xy + y^2 = 0$; (iv.) $x^2 + xy + y^2 = 0$.

Explain the meaning of the tangent being imaginary in the last case.

119. Find the equation of lines joining the origin to the points in which the line $x + 2y = 3$ meets the curve whose equation is

$$x^2 + 2xy - y^2 + 2x + y + 1 = 0.$$

120. If the chord of the circle $x^2 + y^2 = a^2$ whose equation is $lx + my = 1$ subtends an angle $45°$ at the origin, then

$$4\left\{a^2(l^2 + m^2) - 1\right\} = \left\{a^2(l^2 + m^2) - 2\right\}^2.$$

121. Find the coordinates of the six points of intersection of the four lines $x + y + 1 = 0$, $x - y + 2 = 0$, $4x + 2y + 3 = 0$, $x + 2y - 4 = 0$ taken in pairs. Hence find the equations of the three diagonals of the quadrilateral formed by them, and show that the middle points of these diagonals all lie on the straight line whose equation is

$$52x + 80y - 47 = 0.$$

122. Prove that, the axes being rectangular, the area of the triangle formed by the lines $y = mx$, $y = nx$, and $ax + by + c = 0$ is

$$\frac{1}{2}\,\frac{(m \sim n)\,c^2}{(a + bm)\,(a + bn)}.$$

123. Find the value of $h$ so that $x^2 + 2hxy + y^2 + 16x - 5y - 36 = 0$ may represent two straight lines.

124. Show that the equation $6x^2 - xy - 12y^2 - 8x + 29y - 14 = 0$ represents a pair of right lines, and determine their equations.

125. Show that, if a variable straight line cuts off intercepts on two given intersecting straight lines such that the sum of their reciprocals is constant, it must pass through a fixed point.

126. If a straight line move so that the sum of the perpendiculars let fall upon it from two fixed points (3, 4), (7, 2) is equal to three times the perpendicular from a third fixed point (1, 3), show that there is another fixed point through which this line always passes, and find its coordinates.

127. If the equal sides $AB$, $AC$ of an isosceles triangle be produced to $E$, $F$ so that $BE \cdot CF = AB^2$, prove that the line $EF$ will always pass through a fixed point.

128. Prove that the equation

$$x^3 + y^3 - 3x^2y - 3xy^2 + 6x^2 - 6y^2 + 6x + 6y = 0$$

represents three straight lines which pass through a point, and draw a figure showing their positions.

129. Write down the formulæ for turning the axes through the acute angle $\tan^{-1}\frac{3}{4}$.

# *PART II.*

# THE CIRCLE AND CONIC.

# CHAPTER I.

## THE CIRCLE.

**1. To find the equation of a circle—that is, of its circumference—referred to its centre as origin.**

I. *Rectangular axes.*—Let $O$ be the centre, $a$ the radius, $P(x, y)$ any point on the circumference, $PM$ the ordinate

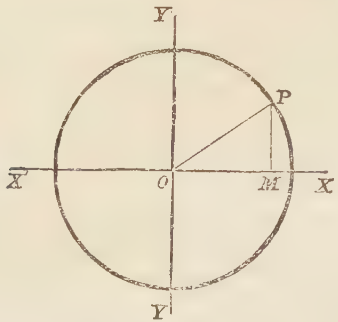



Fig. 1.

of $P$. Since $OP = a$, the geometrical condition satisfied by $P$ is
$$OM^2 + MP^2 = a^2.$$

The analytical expression of this is
$$x^2 + y^2 = a^2 \quad \text{.......................} \quad (1),$$
which is accordingly the equation of the circle.

II. *Oblique axes.*—By similar reasoning to that adopted in Part I., § 2, we see that the equation required is
$$x^2 + y^2 + 2xy \cos \omega = a^2 \quad \text{...............} \quad (2).$$

**2. To find the general equation of the circle (axes rectangular).**

Let $C$ be the centre and let its coordinates be $h$ and $k$. Let $a$ be the radius and $P(x, y)$ any point on the circumference. Draw the ordinates $PM$ and $CN$, and draw $CM'$ parallel to $OX$ to meet $PM$ in $M'$.

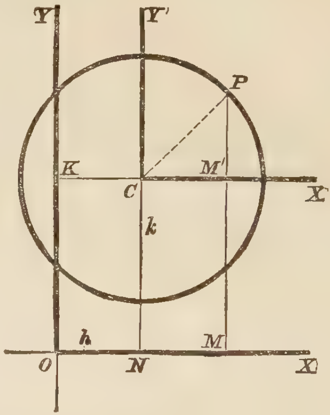



Fig. 2.

The geometrical condition satisfied by $P$ is

$$CM'^2 + M'P^2 = a^2,$$

and the analytical expression of this is

$$(x-h)^2 + (y-k)^2 = a^2 \quad \text{.........} \quad (3).$$

### Exercise.

1. Verify the last equation as follows :—Write down the equation of the circle referred to parallel axes through its centre $C$ in the form $x^2 + y^2 = a^2$, and transfer this equation to the axes $OX$, $OY$. Remember that, since the coordinates of $C$ referred to $OX$ and $OY$ are $(h, k)$, the coordinates of $O$ referred to the parallel axes through $C$ will be $(-h\ -k)$.

3. Equation (3), when expanded, may be written

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - a^2 = 0;$$

so that the general equation of the circle referred to rectangular axes is of the second degree with the coefficients of $x^2$ and $y^2$ equal and the coefficient of $xy$ zero. This general equation is usually written in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...... .....} \quad (4).$$

Rewriting equation (4) in the form

$$(x+g)^2+(y+f)^2 = g^2+f^2-c,$$

and comparing with equation (3), we see that equation (4) represents *a circle whose centre is the point* $(-g, -f)$ and whose radius is $\sqrt{g^2+f^2-c}$.

An alteration in the constant term $c$ in equation (4) will therefore affect the radius, but not the position of the centre; so that by giving $c$ a series of different values we obtain a series of concentric circles.

In the ordinary case the expression $g^2+f^2-c = 0$ will be positive, since it is the square of the radius. When it is zero the equation will represent a circle of zero radius with its centre at the point $(-g, -f)$; that is to say, the point $(-g, -f)$ itself. The locus in this case is known as a **point circle.**

When $g^2+f^2-c$ is negative the quantity under the radical is negative and the circle is wholly imaginary; that is to say, the equation cannot be satisfied by the coordinates of any real points.

4. **Point-circle.** — The explanation of the fact that when $g^2+f^2-c^2 = 0$, the equation represents a point instead of a continuous curve should be noticed. The equation reduces to $(x+g)^2+(y+f)^2 = 0$, which is in reality two equations; for, since a square cannot be negative, each bracket must be separately zero, so that we are, in fact, furnished with the two equations $x+g = 0$ and $y+f = 0$, which determine the point $(-g, -f)$.

5. **To find the general equation of the circle (oblique axes).**

The distance between the points $(x, y)$, $(h, k)$ is the square root of

$$(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega.$$

Therefore the general equation of the circle referred to oblique axes is

$$(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega = a^2,$$

or

$$\boldsymbol{x^2+2xy\cos\omega+y^2-2(h+k\cos\omega)x-2(h\cos\omega+k)y}$$
$$\boldsymbol{+h^2+2hk\cos\omega+k^2-a^2 = 0} \quad \ldots(5);$$

so that in this case *the coefficients of* $x^2$ *and* $y^2$ *are equal, and the coefficient of* $xy$ *is* $(2\cos\omega)$ *times that of* $x^2$ *or* $y^2$.

6. **To find the equation of a circle through three given points.**

*Rectangular axes.*—Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$. This equation involves three constants $g, f, c$, which have to be determined. (Hence, in general, a circle can be found which passes through three given points.) By substituting the coordinates of each of the given points in the above equation, we obtain three simultaneous simple equations in $g, f$, and $c$, from which $g, f$, and $c$ may be determined. It is useless to evolve any formula; all that is necessary is to remember the method.

*Oblique axes.*—The method is similar, but we start with a general equation in which the terms of the second degree are $x^2+y^2+2xy\cos\omega$ instead of $x^2+y^2$.

*Example.*—Find the equation of the circle which passes through the points $(20, 3)$, $(19, 8)$, and $(2, -9)$, the axes being rectangular.

If the equation be $x^2+y^2+2gx+2fy+c=0$, we have, by substitution of coordinates,

$$40g+6f+c=-409,\quad 38g+16f+c=-425,\quad 4g-18f+c=-85,$$

from which $g=-7$, $f=-3$, $c=-111$; so that the equation is

$$x^2+y^2-14x-6y-111=0,$$

or

$$(x-7)^2+(y-3)^2=169,$$

a circle whose centre is at the point $(7, 3)$ and whose radius is 13.

### Exercises.

Nos. 6–9 are for rectangular axes.

2. Find the equations of the circles, having given—

($a$) Centre $(\frac{1}{2}, \frac{1}{2})$, radius $\frac{1}{2}\sqrt{2}$. ($b$) Centre $(3, -2)$, radius 3.

($c$) Centre $(0, -1)$, radius 1.

3. Find the centres and radii of the following circles:—

($a$) $x^2+y^2+4x-4y=1$. ($b$) $x(x+y-6)=y(x-y+8)$.

($c$) $x^2+y^2+2a(x-y)+a^2=0$. ($d$) $(x-a)(x-c)+(y-b)(y-d)=0$.

($e$) $(x-y+a)^2+(x+y-a)^2=2a^2$.

4. Find the angle between the axes when $x^2+xy+y^2-4x-5y-2=0$ represents a circle, and find the radius and coordinates of the centre of this circle.

5. Find the centre and radius of the circle whose equation is $x^2+2xy\cos\omega+y^2+2gx+2fy+c=0$ referred to oblique axes.

6. Find the equations to the circles through the following points, and in each case transform the equations to the centre as orgin :—

(*a*) (1, 0), (2, 3), and (3, −1). (*b*) $(h, k)$, $(h, 0)$, $(0, k)$.

(*c*) (1, 6), (3, 2), (2, 3). (*d*) $(a, b)$, $(a, a)$, $(b, b)$.

7. Find the equation of the circle—

(*a*) Passing through (0, 0), and intercepting lengths $a$ and $b$ on the axes.

(*b*) Whose centre is $(h, k)$ and which passes through the point $(p, q)$

8. Find the locus of a point such that the square of its distance from the origin is equal to its distance from the axis of $x$ multiplied into a constant $a$. Find its centre and radius.

9. If the square of the distance from the origin is $2a$ times the distance from the line $x = \frac{1}{2}a$, show that the locus is a point-circle, and find its position.

10. Prove that the product of the intercepts of the circle

$$(x-x_1)^2+(y-y_1)^2+2(x-x_1)(y-y_1)\cos\omega = r^2$$

on the axes of coordinates are equal, and hence deduce the result of Euclid III. 36.

11. The axes being rectangular, show that the points (5, 5), (6, 4), (−2, 4), and (7, 1) all lie on a circle, and find its centre and radius.

### 7. To find the polar equation of a circle.

If the centre is the pole and the radius $a$, the equation is clearly $r = a$. If the centre $C$ is not at the pole, let

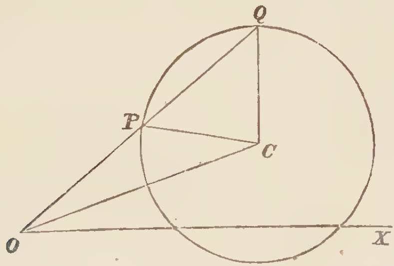



Fig. 3.

its coordinate be $(\rho, \alpha)$. Let $P(r, \theta)$ be any point on the circle. Let $a$ be the radius.

The geometrical condition satisfied by $P$ is

$$OC^2+OP^2-2OC\,.\,OP\cos COP = CP^2,$$

and the analytical expression of this is

$$\rho^2+r^2-2\rho r\cos(\theta-\alpha) = a^2,$$

or, rearranged in descending powers of $r$,

$$r^2-2r\rho\cos(\theta-\alpha)+\rho^2-a^2 = 0 \quad\ldots\ldots\ldots\ (6).$$

**8.** *If the pole is on the circumference,*

$$\rho = a$$

and the equation reduces to

$$r = 2a\cos(\theta-\alpha) \quad\ldots\ldots(7).$$

*If, in addition, the initial line passes through the centre,*

$$\alpha = 0$$

and the equation becomes

$$r = 2a\cos\theta \ \ldots(8).$$

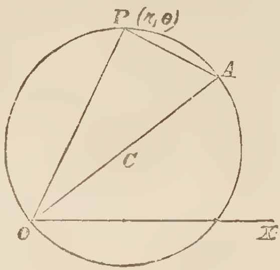



Fig. 4.

9. The lengths $OP$ and $OQ$ are the roots of the above equation (6) in $r$; hence $OP.OQ = \rho^2-a^2$ and so is independent of the direction in which the straight line $OPQ$ is drawn through $O$. Note that when $O$ is within the circle, $OP$ and $OQ$ are of opposite sign; so that the product $OP.OQ$ is negative. (Compare Euclid III. 35 and III. 36.) When $O$ is outside the circle, as in the figure, the radius vector through $O$ will meet the circle in real points only when equation (6) has real roots for $r$; that is, when $\rho^2\cos^2(\theta-\alpha)-\rho^2+a^2$ is positive, or, in other words, when $\rho\sin(\theta-\alpha) < a$, as is geometrically obvious. When $O$ is within the circle $\rho < a$; hence the expression $\rho^2\cos^2(\theta-\alpha)-\rho^2+a^2$ is positive for all values of $\theta$, and any straight line through $O$ will meet the circle in real points.

**Exercise.**

12. Find the centre and radius of (*a*) $r^2-6r\cos(\theta-\frac{1}{5}\pi)-16 = 0$, (*b*) $r^2-2r(\sqrt{3}\cos\theta+\sin\theta) = 5$.

10. **Chords and tangents.** DEFINITIONS.—A **chord** of a curve is any straight line joining two points on the curve. A straight line intersecting the curve is also sometimes called a secant to the curve.

The student accustomed to Euclid's definition of a tangent must now familiarize himself with a fresh conception and a fresh definition of a tangent—applicable not only to circles, but to curves in general. The tangent to a curve at any point is in reality nothing more than the straight line passing through the point and indicating the direction of the curve at that point. To illustrate this conception, consider the recently constructed "topsy-turvy railway" at the Crystal Palace. The tangent to the curve formed by the railway at any point $P$ is the direction in which the wheels of the car are moving as they pass through $P$—that is, the direction in which the wheels would leave the curve and start off through space if the rails were suddenly withdrawn as the wheels passed through $P$.

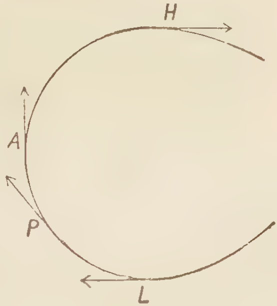


Fig. 5.

At the lowest point $L$ this direction is horizontal. At the highest point $H$ it is horizontal and in the opposite direction, while at some intermediate point $A$ on the ascent it is vertically upwards.

We have now explained that the tangent to a curve at any point gives the direction of the curve at that point. This direction can only be identified by means of a straight line through the point; for a straight line maintains its direction unaltered for a finite distance, whereas a curve (using the word in the popular sense) takes a new direction as soon as we travel any distance, however short, along it. Hence we must conceive of the tangent to a curve at $P$ as the straight line drawn through $P$ which marks the direction of the curve at $P$. Now the

only intelligible interpretation of the direction of the curve at $P$ is the direction in which we travel from $P$ to the next consecutive point $P'$ on the curve—that is, the direction of the straight line joining $P$ to the next consecutive point $P'$ on the curve. The most convenient way of arriving at this direction is to take a chord $PQ$, and to cause the point $Q$ to move along the curve until it reaches the point $P'$, which is the next consecutive point to $P$—that is, until it is on the point of coinciding with $P$; then the direction of the curve at $P$ is the limiting position to which the direction of the chord tends as the second point of contact $Q$ moves up to and tends to coincide with $P$.

NOTE.—The student will observe that when $Q$ actually coincides with $P$ the direction $PQ$ becomes indeterminate. To be strictly accurate, then, we must say that when the point $Q$ moves up to and is only an infinitesimal distance from the point $P$ the direction of $PQ$ moves up to and is only inclined at an infinitesimal angle to its "limiting position when $Q$ coincides with $P$."

With the above introduction and the above qualification the student will be able to understand the following:—

11. DEFINITION.—"The **tangent to a curve** at any point is the limiting position of the chord of the curve drawn through that point when the second point of intersection moves up to and coincides with the first."

NOTE.—The tangent, being a straight line, must be considered as infinite in both directions, and in the case of any continuous curve we shall obtain the same infinite straight line for the tangent at any point, in whichever direction we travel along the curve from that point.

12. If the coordinates of a point $P$ on the curve be $x_1$, $y_1$, and those of another point $Q$ on the curve $x_2$, $y_2$, the equation of the chord $PQ$ is $\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$ (Part I., § 10 C), and the problem of finding the equation of the tangent at $P$ is simply that of finding the limit to which the expression $(y_2-y_1)/(x_2-x_1)$ tends as the point $(x_2, y_2)$, moving along the curve, approaches and tends to coincide with $(x_1, y_1)$, remembering that the coordinates of both points must satisfy the equation of the curve.

**13. To find the equation of the tangent at the point $(x_1, y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$.**

Let $P$ be the point $(x_1, y_1)$, $Q$ a point $(x_2, y_2)$ on the circle near to $P$.

The equation of $PQ$ is

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

(Pt. I., § 10 C)

Now, since $P$ and $Q$ are on the circle,

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0$$
$$x_2^2+y_2^2+2gx_2+2fy_2+c=0.$$

Subtracting

$$x_2^2-x_1^2+y_2^2-y_1^2+2g(x_2-x_1)+2f(y_2-y_1)=0,$$

or $\quad (x_2-x_1)(x_1+x_2+2g)+(y_2-y_1)(y_1+y_2+2f)=0,$

$$\therefore \quad \frac{y_2-y_1}{x_2-x_1}=-\frac{x_1+x_2+2g}{y_1+y_2+2f} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a)$$

Hence **the equation of the chord $PQ$ is**

$$\boldsymbol{(x-x_1)(x_1+x_2+2g)+(y-y_1)(y_1+y_2+2f)=0 \ \ldots \ (9).}$$

We have now obtained the equation of the chord in a form which does not become indeterminate when $(x_2, y_2)$ moves up to and coincides with $(x_1, y_1)$.

Putting $x_2=x_1$ and $y_2=y_1$, we have for the equation of the tangent at $(x_1, y_1)$

$$(x-x_1)(x_1+g)+(y-y_1)(y_1+f)=0,$$

*i.e.* $\quad x(x_1+g)+y(y_1+f)=x_1^2+y_1^2+gx_1+fy_1,$

or, since $x_1^2+y_1^2+2gx_1+2fy_1+c=0$, the equation of the tangent may be written

$$\boldsymbol{x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0 \ \ldots \ (10).}$$

This equation may also be written in the form

$$\boldsymbol{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0 \ \ldots \ (11),}$$

and so may be derived from the equation of the circle by writing $xx_1$ for $x^2$, $yy_1$ for $y^2$, $x+x_1$ for $2x$, and $y+y_1$ for $2y$.

**Exercises.**

13. By a similar method show that the equation of the tangent at the point $(x_1, y_1)$ to the curve $ax^2+2hxy+by^2+2gx+2fy+c=0$ is
$$x(ax_1+hy_1+g)+y(hx_1+by_1+f)+gx_1+fy_1+c=0.$$

14. Find the equation of the tangent to $x^2+y^2-4x-4y+4=0$ at the point (2, 4).

15. Find the equations of the tangents to the circle $x^2+y^2=13$ at the points where $x=2$.

16. Find the equations of the tangents $x^2+y^2-2x-2y=23$ at the points where it is cut by the circle $x^2+y^2=25$.

17. Show that the circles $x^2+y^2=25$ and $x^2+y^2+8x+6y-75=0$ touch each other.

18. Show that the equation of the secant joining points $(x_1, y_1)$ and $(x_2, y_2)$ on the circle $x^2+y^2=a^2$ may be written
$$x(x_1+x_2)+y(y_1+y_2)=x_1x_2+y_1y_2+a^2.$$

19. If the equation of the circle is $x^2+y^2+2gx+2fy+c=0$, then the equation of the secant is
$$x(x_1+x_2+2g)+y(y_1+y_2+2f)=x_1x_2+y_1y_2-c.$$

20. If the tangents at $(x_1, y_1)$ and $(x_2, y_2)$, points on the circle
$$x^2+y^2+2gx+2fy+c=0.$$
are at right angles, prove the relation
$$x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+g^2+f^2=0.$$

21. The tangent at $P$ to the circle $x^2+y^2=a^2$ meets the axes of $x$ and $y$ respectively in $T$ and $t$, and $PM$, $PN$ are drawn perpendicular on these axes: prove that $CM \,.\, CT=a^2$, and that $CN \,.\, Ct=a^2$. $C$ being the centre of the circle.

14. **Normal.** DEFINITION.—The **normal** at any point of a curve is the straight line through that point perpendicular to the tangent.

15. **To find the equation of the normal at the point $(x_1, y_1)$ to the circle $x^2+y^2+2gx+2fy+c=0$.**

The form of the equation to the circle shows the axes to be rectangular.

The normal is the straight line through $(x_1, y_1)$ perpendicular to $x(x_1+g)+y(y_1+f)+(gx_1+fy_1+c)=0$.

Hence the equation of the normal is
$$\frac{x-x_1}{x_1+g}=\frac{y-y_1}{y_1+f}. \quad \text{(Pt. I., § 10 B and § 19)}$$

Hence the normal passes through the centre $(-g, -f)$, and therefore coincides with the radius drawn to the point $(x_1, y_1)$. Hence the tangent to a circle is at right angles to the radius.

[The student must remember that we are investigating tangent properties of the circle *ab initio*, and therefore Euclid's propositions must not be assumed.]

**Exercise.**

22. Show that the equation of the normal at $(x_1, y_1)$ to the curve $ax^2+2hxy+by^2+2gx+2fy+c=0$ the axes being rectangular, is

$$\frac{x-x_1}{ax_1+hy_1+g}=\frac{y-y_1}{hx_1+by_1+f}.$$

16. By performing the above operations on the equation of the circle in its simplest form $x^2+y^2=a^2$, or by the direct substitutions $g=0$, $f=0$, $c=-a^2$ in the above equations, we find that the equation of the tangent at $(x_1, y_1)$ to the circle $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$; and that of the normal through the same point is $x/x_1=y/y_1$ the normal of course passing through the origin, which is now the centre of the circle.

**Exercises.**

23. Find the equation of the normal at the point (5, 12) to the circle $x^2+y^2=13$.

24. Find the equation of the normal at the point (5, 2) to the circle $x^2+y^2-5x-4y+4=0$.

17. **To find the coordinates of the points of intersection of a straight line and a circle whose equations are given.**

All we have to do is to solve the two equations for $x$ and $y$. (Pt. I., § 9.)

*Example.*—Find the coordinates of the point of intersection of the straight line $7x-y-25=0$ and the circle $x^2+y^2=25$.

Let $x$, $y$ be the coordinates required.
From the equation of the straight line we get $y=7x-25$.
Substituting in the equation of the circle we have

$$x^2+(7x-25)^2=25.$$

From which $x=3$ or $4$, $y=-4$ or $3$.
Hence the coordinates are $(3, -4)$, and $(4, 3)$.

18. The result of the operation in § 17 is clearly to give a quadratic in $x$ for the abscissa required. This quadratic will have two roots; corresponding to each root there will be a single value of $y$ derived by substitution in the equation of the straight line.

Hence a straight line will cut a circle in two points, which will be real if the roots of the quadratic are real, and imaginary if the roots of the quadratic are imaginary, in which latter case the straight line will lie wholly outside the circle.

If the roots of the quadratic are equal, the straight line cuts the circle in two coincident points, and is therefore the limiting position of a chord when the points of intersection become coincident—that is to say, it is a tangent to the circle. Hence the condition that the straight line should touch the circle is that the quadratic obtained by solving for $x$ or $y$ should have equal roots.

One result should be remembered—namely, the condition that the straight line $y = mx + b$ should touch the circle $x^2 + y^2 = a^2$.

Solving for $x$, we have

$$x^2 + (mx+b)^2 = a^2 \quad \text{or} \quad x^2(1+m^2) + 2mbx + b^2 - a^2 = 0.$$

The condition that this should have equal roots is

$$(1+m^2)(b^2-a^2) = m^2b^2 \quad \text{or} \quad b^2 = a^2(1+m^2).$$

Hence the straight lines $y = mx \pm a\sqrt{1+m^2}$ touch the circle $x^2+y^2 = a^2$ for all values of $m$.

## Exercises.

25. Show that the circle $x^2+y^2+2a(x+y)+a^2 = 0$ touches the axes of coordinates, and find the points of contact.

Find the equations of the circles whose centres are at the origin and which touch respectively:

26. $y = \frac{1}{3}x + 3\frac{1}{3}$. 27. $3x + 4y = 10$.

28. Show that the following straight lines and circles touch, and determine the points of contact in each case:—

(a) $x^2+y^2+x+y = 0$ and $x+y+2 = 0$.

(b) $x^2+y^2 = 6y$ and $y = x\sqrt{3}+9$.

29. Show that the line $y = mx + a\left\{1 \pm \sqrt{(1+m^2)}\right\}$ is always a tangent to the circle $x^2+y^2 = 2ay$.

**19 To find the condition that the straight line $lx+my+n=0$ should touch the circle**

$$x^2+y^2+2gx+2fy+c=0.$$

The equation of the lines joining the origin to the points of intersection of the straight line and circle is

$$n^2(x^2+y^2)-2n(lx+my)(gx+fy)+c(lx+my)^2=0 \ldots (A).$$

If the straight line touches the circle, it meets it in coincident points. Equation (A) therefore represents a pair of coincident lines in such a case, and must be a perfect square. The condition for this is

$$(cl^2-2gnl+n^2)(cm^2-2fmn+n^2) = (clm-gmn-fnl)^2,$$

which reduces to

$$c(l^2+m^2)+n^2-(fl-gn)^2-2fmn-2gnl=0 \ \ldots (B).$$

20. Note.—The last equation is unsymmetrical. It is useless to remember the result—the method only should be remembered. As an example the student should prove by the same method that the condition that the straight line $lx+my+n=0$ should touch the curve

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is $(bc-f^2)l^2+(ca-g^2)m^2+(ab-h^2)n^2+2(gh-af)mn+2(hf-bg)nl$
$+2(fg-ch)lm=0$ (C).

Equation (C) is symmetrical, and equation (B) is the reduced form of this symmetrical equation when $a=1$, $b=1$, $h=0$.

### Exercises.

30. Find the equations of the tangents to the circle $x^2+y^2+2x=0$ which make 45° (measured counterclockwise) with the axis of $x$.

31. Find the equations of the tangents to the circle $x^2+y^2=25$ which are parallel to $3x+4y=0$.

32. Find the equations of the tangents to the circle $x^2+y^2=2$ which are inclined to the axis of $x$ at (a) 120°, (b) −30°, (c) $\tan^{-1}\frac{5}{12}$.

33. Write down the equation of the pair of straight lines joining the origin to the intersections of the line $x/a+y/b=1$ and the circle $x^2+y^2=c^2$.

Show that, if the line touches the circle, then $1/a^2+1/b^2=1/c^2$.

34. Show that the straight lines joining the origin to the points of intersection of the circles $x^2+y^2=a^2$ and $x^2+y^2+2(gx+fy)=0$ are given by the quadratic equation $a^2(x^2+y^2)-4(gx+fy)^2=0$.

35. Find the points of intersection of the circles $x^2+y^2=25$ and $x^2+y^2-26x+25=0$, and show that the circles cut at right angles.

## MISCELLANEOUS EXERCISES ON CHAP. I.

36. Find the locus of a point which moves so that the sum of the squares of its distances from $(a, 0)$ and $(-a, 0)$ is equal to $2b^2$.

37. Find the points of intersection of the circles $x^2+y^2=9$ and $x^2+y^2-10x+9=0$. Find the equations of the tangents to both circles at the points of intersection.

Show that the two tangents at either point are at right angles.

38. Prove generally that the circles

$$x^2+y^2=a^2 \quad \text{and} \quad x^2+y^2+2gx+a^2=0$$

cut at right angles.

39. Find the equations of the tangents to the circle

$$x^2+y^2+2ax+2by+c=0$$

which are parallel to the line $3x+4y+7=0$.

40. Find the equations of the two tangents to the circle $x^2+y^2=25$ which make an angle of 30° with the axis of $x$.

41. Show by any method that, for all values of $\alpha$, the straight line $r\cos(\theta-\alpha)=\rho\cos\alpha+a$ touches the circle $r^2-2r\rho\cos\theta+\rho^2-a^2=0$.

42. Find the points of intersection of the circle $r=4\cos\theta$ with the straight line $r\cos\theta=3$. Show that these points together with the origin form an equilateral triangle.

43. Show that the square of the distance between the two points $(x_1, y_1)$ and $(x_2, y_2)$ on the circle $x^2+y^2=a^2$ is equal to

$$2(a^2-x_1x_2-y_1y_2).$$

44. Deduce that the equation of a straight line meeting the circle in two points at equal distances $d$ from a point $(x_1, y_1)$ on the circumference is $xx_1+yy_1-a^2+\frac{1}{2}d^2=0$, and apply this result to find the equation of the tangent at $(x_1, y_1)$.

45. Find the equation with oblique axes of the circle that passes through the origin and makes intercepts $p$, $q$ on the axes.

46. Find the condition that the straight line $ax+by=1$ may touch the circle $r=k\cos\theta$.

47. Find the length of the radius of the circle passing through the points $(0, 0)$, $(a, \alpha)$, $(b, \beta)$.

48. Prove from the equation of the circle that the angles in the same segment of a circle are equal.

# CHAPTER II.

## THE CIRCLE (*continued*).

**21. To find the equation of the chord of contact of tangents drawn to a circle from a given external point.**

Let $P$ be the given external point, $PQ$ and $PR$ the tangents to the circle from $P$.

For simplicity we shall work with the equation of the circle referred to rectangular axes with the centre for origin.

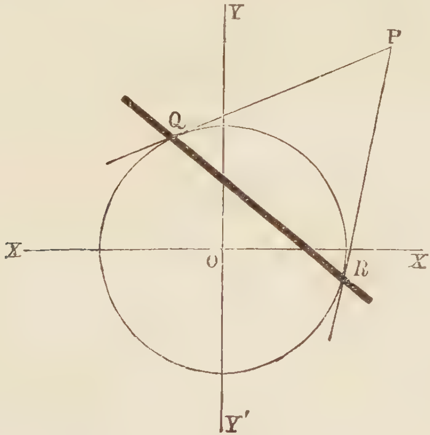



Fig. 6.

Let the coordinates of $P$ be $x_1$, $y_1$, and those of $Q$ and $R$ $h'$, $k'$ and $h''$, $k''$ respectively. (It should be observed that the coordinates of $Q$ and $R$ have no place in the final result.) The equation of $QP$, the tangent at $Q$, is

$$xh' + yk' = a^2.$$

The equation of $RP$ is similarly $xh''+yk''=a^2$.

Since both these lines pass through $(x_1, y_1)$, we have

$$x_1h'+y_1k'=a^2 \quad \text{and} \quad x_1h''+y_1k''=a^2.$$

Hence the coordinates $(h', k')$ and $(h'', k'')$ both satisfy the equation $xx_1+yy_1=a^2$, that is to say, the points $Q$ and $R$ lie on the straight line

$$xx_1+yy_1=a^2 \text{ .................... } (1),$$

which is accordingly the equation of $QR$.

When $P$ is within the circle, no tangents can be drawn from $P$ to the circle. We must therefore search for a fresh geometrical interpretation of equation (1).

(i.) It will be observed that, if the chord of contact $QR$ passes through a fixed point $(h, k)$, within or without the circle, the coordinates $(h, k)$ satisfy equation (1), and accordingly we have $x_1h+y_1k=a^2$; that is, the point $P$ lies on a fixed straight line whose equation is $xh+yk=a^2$. This at once supplies us with a geometrical interpretation of the equation $xh+yk=a^2$ which is independent of the position of the point $(h, k)$ relative to the circle, and it will be seen that, if chords of a circle be drawn through a fixed point, the locus of the intersection of the pairs of tangents drawn to the circle at the extremities of the chords will be a straight line. This straight line is called the **polar** of the fixed point, and the equation $xh+yk=a^2$ represents the polar of the point $(h, k)$ with respect to the circle $x^2+y^2=a^2$. The polar of an external point is the chord joining the points of contact of the tangents drawn to the circle from the given external point.

(ii.) Further, it will observed that, if $P$ $(x_1, y_1)$ lies on the fixed straight line $xh+yk=a^2$ (whether this line does or does not cut the circle in real points) we have

$$x_1h+y_1k=a^2,$$

and accordingly the straight line $QR$ whose equation is $xx_1+yy_1=a^2$ passes through the fixed point $(h, k)$. Hence, if from points lying on a fixed straight line pairs of tangents be drawn to a circle, the chords joining their points of contact will pass through a fixed point. This point is

called the **pole** of the given straight line, and we see that the co-ordinates of the pole of the straight line $xh+yk=a^2$ with regard to the circle $x^2+y^2=a^2$ are $(h, k)$. The pole of a straight line which cuts a circle in real points is the point of intersection of the tangents at the points where the straight line cuts the circle.

(iii.) The pole of the straight line $xh+yk=a^2$ with respect to the circle $x^2+y^2=a^2$ is the point $(h, k)$. Hence the pole of the straight line $x\cos\alpha+y\sin\alpha=p$ is the point $(a^2\cos\alpha/p,\ a^2\sin\alpha/p.)$ If $P$ be the pole, $QR$ the polar, $O$ the centre, it is evident that $OP$ is the direction of the perpendicular from $O$ on $QR$. If $OP$ cut $QR$ in $P'$, $OP'=p$, since $OP'$ is the perpendicular from the origin on $x\cos\alpha+y\sin\alpha=p$.

Further, $OP=\sqrt{(a^2\cos\alpha/p)^2+(a^2\sin\alpha/p)^2}=a^2/p$, and hence $OP.OP'=a^2$.

If $P$ is an external point, this result is geometrically obvious, as follows:—Let $PQ$ be a tangent from $P$; then $OP'Q$ and $OQP$ are similar right-angled triangles, and therefore $OP':OQ=OQ:OP$.

*Example.*—If $OP$ cut the circle in $A$ and $B$, show that

$$1/PA+1/PB=2/PP'.$$

**Exercises.**

1. Write down the polars of the following points with respect to the circle $x^2+y^2=14$ :—

(*a*) $(21, -35)$. (*b*) $(-3, 1)$. (*c*) $(0, 1)$.

2. Find the poles of the following lines with respect to the circle $x^2+y^2=14$ :— (*a*) $3x-y=2$. (*b*) $x-y=14$. (*c*) $3x=7$.

3. Find the pole of the line $lx+my=1$ with respect to the circle $x^2+y^2=a^2$.

**22. If the polar of $P$ passes through $T$, then the polar of $T$ will pass through $P$.**

Let the coordinates of $P$ be $(x_1, y_1)$, those of $T$ $(x_0, y_0)$, and let the equation of the circle be $x^2+y^2=a^2$.

The equation of the polar of $P$ is $xx_1+yy_1=a^2$. Since the point $(x_0, y_0)$ is on this line, we have $x_0x_1+y_0y_1=a^2$.

Hence the point $(x_1, y_1)$ lies on the line $xx_0+yy_0=a^2$; that is, $P$ lies on the polar of $T$.

23. **General equation of polar.**

We have worked above with the simplest form of the equation to the circle, so that the student's attention may be diverted as little as possible from the reasoning involved. Similar processes will show that the point $(x_1, y_1)$ and the straight line

$$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0$$

are respectively pole and polar with regard to the circle

$$x^2+y^2+2gx+2fy+c=0.$$

NOTE.—It must be carefully noticed that **the equation of the polar is of exactly the same *form* as that of the tangent,** and thus need not be remembered separately. *The important difference between the two lines is that, whereas in the case of the tangent $(x_1, y_1)$ was on the curve, no such restriction is now imposed.*

It follows, of course, that

*when a point is* ON *the curve its polar is the same as its tangent.*

This is also clear from geometrical considerations, because, as the point $P$ gradually approaches the curve, the points of contact $Q$ and $R$ become closer and closer together, and the line joining them ultimately becomes a tangent to the curve when $T$ is on the curve. Thus the tangent is only a particular case of the polar.

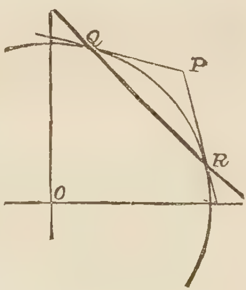



Fig. 7.

### Exercises.

4. Find the polar of the point (5, 4) with respect to the circle

$$x^2+y^2-4x-3y=8.$$

5. **The polar of the origin with respect to the circle**

$$x^2+y^2+2gx+2fy+c=0$$

**is the line** $$gx+fy+c=0.$$

24. **If a straight line drawn through a given point $O$ is cut by the circle in the points $A$ and $B$ and by the polar of $O$ with regard to the circle in the point $O'$, then**

$$\frac{1}{OA}+\frac{1}{OB}=\frac{2}{OO'}.$$

Take $O$ for origin with the axes rectangular, and let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$.

Let $OA=r_1$, $OB=r_2$, $OO'=\rho$. Let the straight line $OAB$ be inclined at an angle $\theta$ to the axis of $x$.

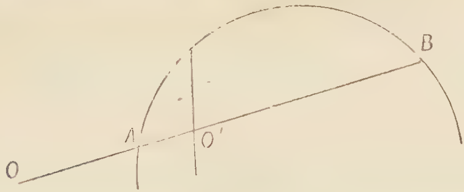



Fig. 8.

Then $r_1$, $r_2$ are the two values of the radius vector $r$, for which the Cartesian coordinates $(r\cos\theta, r\sin\theta)$ satisfy the equation of the circle. Writing $r\cos\theta$ for $x$ and $r\sin\theta$ for $y$ in the equation of the circle, we see that $r_1$ and $r_2$ are the roots of the equation

$$r^2+2r(g\cos\theta+f\sin\theta)+c=0.$$

Hence

$$\frac{1}{r_1}+\frac{1}{r_2}=\frac{r_1+r_2}{r_1r_2}=-\frac{2(g\cos\theta+f\sin\theta)}{c}.$$

Similarly, $\rho$ is the value of $r$ for which the coordinates $(r\cos\theta, r\sin\theta)$ satisfy the equation $gx+fy+c=0$, that is, the value of $r$ for which $r(g\cos\theta+f\sin\theta)+c=0$.

Hence $$\frac{1}{\rho}=-\frac{g\cos\theta+f\sin\theta}{c};$$

$$\therefore\quad \frac{1}{r_1}+\frac{1}{r_2}=\frac{2}{\rho}\quad\text{or}\quad\frac{1}{OA}+\frac{1}{OB}=\frac{2}{OO'}.$$

This result is usually expressed by stating that any straight line through a given point $O$ is "cut harmonically" by any circle and the polar of $O$ with regard to that circle.

**25. To find the lengths of the tangents drawn from a given point to a given circle.**

Let $(x', y')$ be the given point $P$, and let the equation of the given circle be $x^2+y^2+2gx+fy+c=0$.

If $PAB$ be a straight line drawn through $P$ at an angle $\theta$ to the axis of $x$, cutting the circle in $A$ and $B$, the lengths $PA$ and $PB$ will (by Pt. I., § 10 B, Cor.) be the values of $r$ for which the coordinates $(x'+r\cos\theta,\ y'+r\sin\theta)$ satisfy the equation of the circle—that is to say, they will be the roots of the quadratic in $r$

$$(x'+r\cos\theta)^2+(y'+r\sin\theta)^2+2g(x'+r\cos\theta)$$
$$+2f(y'+r\sin\theta)+c=0,$$

or $$r^2+2r\{(x'+g)\cos\theta+(y'+f)\sin\theta\}$$
$$+x'^2+y'^2+2gx'+2fy'+c=0 \quad \text{(A)}.$$

Hence $$PA\,.\,PB=x'^2+y'^2+2gx'+2fy'+c \quad\ldots\ldots\ldots \text{(B)},$$
and so is independent of the direction in which the chord is drawn through $P$. (*Cf.* Euc. III. 36.)

When the straight line drawn through $P$ touches the circle $A$ and $B$ are coincident, and $PA$ and $PB$ become each equal to the tangent from $P$ to the circle; hence the **square of the length of this tangent is**

$$\boldsymbol{x'^2+y'^2+2gx'+2fy'+c} \quad\ldots\ldots\ldots\ldots\ldots \textbf{(2)}$$

—that is, the result of substituting the coordinates of $P$ in the left-hand member of the equation of the circle (written with the coefficients of $x^2$ and $y^2$ each unity).

[The expression $x'^2+y'^2+2gx'+2fy'+c$ may be written $(x'+g)^2+(y'+f)^2-(g^2+f^2-c)$, and so is equal to the square of the distance of the given point from the centre of the circle less the square of the radius.

*Note.*—To impress the method on his memory, the student should follow out the corresponding process with regard to the curve defined by the general equation $ax^2+2hxy+by^2+2gx+2fy+c=0$.

If $PAB$, $PA'B'$ be chords drawn through $P(x', y')$ at inclinations $\theta$, $\theta'$ to the axis of $x$ to meet the curve in $A$ and $B$, $A'$ and $B'$, it will be found that

$$PA\,.\,PB=\frac{ax'^2+2hx'y'+by'^2+2gx'+2fy'+c}{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta}$$

and $$PA'\,.\,PB'=\frac{ax'^2+2hx'y'+by'^2+2gx'+2fy'+c}{a\cos^2\theta'+2h\cos\theta'\sin\theta'+b\sin^2\theta'};$$

so that the ratio $PA \,.\, PB : PA' \,.\, PB'$ depends only on the directions of the intersecting chords, and not on the position of the point of intersection. [See § 132].

If the point $P$ is within the circle, the expression $x'^2+y'^2+2gx'+2fy'+c=0$ will be negative (for $PA$ and $PB$ are now of opposite sign) and will represent the rectangle under the lengths $AP$ and $PB$ with the minus sign prefixed.

### Exercises.

6. Find the lengths of the tangents to (a) $x^2+y^2=20$ from the point $(6, 3)$; (b) $(x-1)^2+(y-2)^2=4$ from the origin; (c) $2x^2+2y^2-5x-7y+6=0$ from $(2, 3)$.

7. Show that the difference of the squares of the tangents drawn from a point to two concentric circles is independent of the position of the point.

26. **To find the locus of the middle points of a system of parallel chords of a circle.**

Let the chords be inclined at an angle $\theta$ to the axis of $x$, and let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots\ldots\ldots\ldots\ldots \quad (1).$$

Let $(x', y')$ be the middle point of one of the chords; then the two values of $r$ for which the coordinates $(x'+r\cos\theta,\ y'+r\sin\theta)$ satisfy equation (1) must be equal in magnitude and opposite in sign, since the two halves of the chord are drawn in opposite directions from $(x', y')$; hence the sum of the roots of equation (A) § 25, must be zero—that is,

$$(x'+g)\cos\theta+(y'+f)\sin\theta=0.$$

Hence the locus required is the straight line

$$(x+g)\cos\theta+(y+f)\sin\theta=0 \quad \ldots\ldots\ldots \quad (3)$$

which passes through the centre $(-g, -f)$, and is perpendicular to the parallel chords, since the perpendicular from the origin on the line (3) is inclined at an angle $\theta$ to the axis of $x$. (*Cf.* Euc. III. 3.)

### Exercise.

8. Find the locus of the middle points of chords of the circle $x^2+y^2=13$ which are parallel to $3x+4y-5=0$.

**27. To find the equation of the common chord of the two circles $x^2+y^2+2g_1x+2f_1y+c_1=0$ and $x^2+y^2+2g_2x+2f_2y+c_2=0$.**

The equation

$$(x^2+y^2+2g_1x+2f_1y+c_1)-(x^2+y^2+2g_2x+2f_2y+c_2)=0$$

reduces to

$$2(g_1-g_2)x+2(f_1-f_2)y+(c_1-c_2)=0 \ldots\ldots (4),$$

and is therefore the equation of some straight line. Further, it is satisfied by any values of $x$ and $y$ which satisfy the equations of both circles. Hence equation (4) is the equation of the straight line which passes through the points of intersection of the circles—that is, the equation of the common chord. When the circles do not intersect in real points equation (4) still represents some straight line. We shall find out the geometrical interpretation of this straight line in § 29.

**28. Radical axis.**—Definition. The **radical axis** of two circles is the locus of points from which the tangents to the two circles are equal in length.

**29. To find the equation of the radical axis of two circles.**

If $(x', y')$ be a point on the radical axis, and the equations of the circles as above, then we have, by § 25,

$$x'^2+y'^2+2g_1x'+2f_1y'+c_1=x'^2+y'^2+2g_2x'+2f_2y'+c_2$$

or

$$2(g_1-g_2)x'+2(f_1-f_2)y'+(c_1-c_2)=0.$$

Hence the equation of the radical axis of the two circles is

$$2(g_1-g_2)x+2(f_1-f_2)y+(c_1-c_2)=0 \ldots\ldots (5).$$

This shows that equation (B), in § 27, is the equation of the radical axis of the two circles, an interpretation which is geometrically intelligible whether the circles do or do not intersect in real points.

The identity of the equations obtained for the common chord and radical axis shows that the radical axis of two intersecting circles is their common chord, or, in other words, that the tangents to the two circles from any point on their common chord are equal. Geometrically, this follows at once from Euclid III. 36. In particular, the common chord bisects the common tangents.

It will be observed since the equation of the line joining the centres $(-g_1, -f_1)$, $(-g_2, -f_2)$ is

$$(x+g_1)/(g_1-g_2) = (y+f_1)/(f_1-f_2),$$

that the radical axis is perpendicular to the line joining the centres.

30. In abridged notation (see Part I., § 27), if $S'$ and $S''$ be used to denote the expressions $x^2+y^2+2g_1x+2f_1y+c_1$ and $x^2+y^2+2g_2x+2f_2y+c_2$, the equation of the radical axis of the circles $S'=0$ and $S''=0$ may be written $S'-S''=0$; but it must be carefully remembered that the use of this abridgment assumes that all the terms in the equation of the circle have been carried to the left-hand side and the coefficients of $x^2$ and $y^2$ reduced to unity before we adopt the abridged symbol.

31. **General equation of circles having the same radical axis as two given circles.**

The general equation of all circles having a common radical axis will be $S'+\lambda S''=0$, where $S'=0$ and $S''=0$ are the equations of any two circles of the system.

For the radical axis of $S'=0$ and $S''=0$ is $S'-S''=0$; and, since the coefficient of $x^2$ or $y^2$ in the expression $S'+\lambda S''$ is $1+\lambda$, the radical axis of $S'=0$ and $S'+\lambda S''=0$ is (remembering the caution at the end of the last paragraph) $S'-(S'+\lambda S'')/(1+\lambda)=0$ or $S'-S''=0$; that is, the same as the radical axis of the two given circles.

Now the equation $S'+\lambda S''=0$ is satisfied by the coordinates of any point which satisfies both the equations $S'=0$ and $S''=0$. When, therefore, the two points of intersection of the circles $S'=0$ and $S''=0$ are real, all the circles pass through the same pair of real points.

When $S'=0$ and $S''=0$ do not meet in real points we can still solve their equations simultaneously and obtain two imaginary values of $x$ with two corresponding imaginary values of $y$, which we may regard as the coordinates of two imaginary points satisfying the equations $S'=0$ and $S''=0$ and therefore the equation $S'+\lambda S''=0$. Hence we may say that in this case all the circles of the system pass through the pair of imaginary

points which are the intersections of $S'=0$ and $S''=0$, though we cannot represent these points on a figure. It is clear that the centres of a system of coaxal circles—that is, circles having a common radical axis—are collinear, for the line joining any two centres is perpendicular to the common radical axis.

32. **The three radical axes of three circles taken in pairs meet in a point.**

Let the equations of the circles in abridged notation be $S'=0$, $S''=0$, $S'''=0$.

The equations of the three radical axes are

$$S'-S''=0,$$
$$S''-S'''=0,$$
$$S'''-S'=0.$$

Since the three left-hand members of the equations when added together vanish identically, the three radical axes meet in a point. (Pt. I., § 23.)

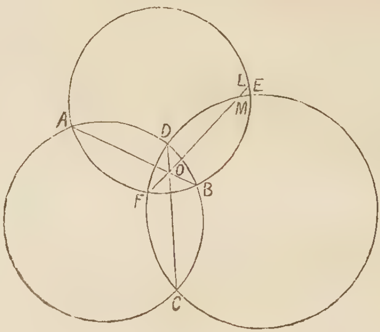



Fig. 9.

This point is known as the "radical centre" of the three circles.

**Exercises.**

9. Find the equation of the common chord of the circles

$$x^2+y^2-4x+4y-1=0 \quad \text{and} \quad x^2+y^2+6x-3y-1=0.$$

10. Show that the circles $x^2+y^2+2y=0$, $x^2+y^2+x-y-2=0$, $x^2+y^2+3x-7y-6=0$ have a common radical axis, and find it.

11. Find the equation of the radical axis of the circles

$$x^2+y^2+2x+4y+4=0 \quad \text{and} \quad 4x^2+4y^2-24x+8y+25=0.$$

12. Find the equation of the radical axis of the two circles

$$x^2+y^2+2x+4y=7 \quad \text{and} \quad x^2+y^2-6x+2y=5,$$

and show that it is at right angles to the line joining the centres of the two circles.

13. Find the radical centre of the circles $x^2+y^2-3x-6y+8=0$, $x^2+y^2-x-4y+2=0$, $x^2+y^2+2x-6y+3=0$.

33. **Limiting points.**—By giving a suitable value to $\lambda$ in the equation $S' + \lambda S'' = 0$ we can make the radius of the circle $S' + \lambda S'' = 0$ zero, that is, obtain a point-circle as one of the circles of a coaxal system. We shall find that the equation giving the necessary value or values of $\lambda$ will be a quadratic, and so there are in general two point-circles (real or imaginary) which are members of a coaxal system. These are called the "limiting points" of the system.

**To show that the limiting points of a coaxal system are imaginary when the circles of the system intersect in real points, and real when the circles intersect in imaginary points.**

The first part of this theorem is geometrically obvious; for no point-circle could pass through two points at a finite distance from each other.

To prove both parts analytically, take the line joining the centres of the coaxal circles for axis of $x$ and the radical axis for axis of $y$. (If we take the general equation of the circle to work with in this instance, it will be found that the algebra becomes too complicated for convenience.)

Let $(h, 0)$ be the centre and $a$ the radius of a circle of the system.

Its equation is $\qquad (x-h)^2 + y^2 = a^2$ ............................ (A).

The equation of the radical axis is $x = 0$ ............................ (B).

Any circle of the system passes through the points of intersection (real or imaginary) of (A) and (B); hence the equation of any circle of the system is of the form

$$(x-h)^2 + y^2 - a^2 + \lambda x = 0 \quad \text{.........................} \quad (C)$$

or
$$x^2 + y^2 + (\lambda - 2h)x + h^2 - a^2 = 0.$$

If this circle has zero radius, we have

$$(\tfrac{1}{2}\,\overline{\lambda - 2h})^2 - (h^2 - a^2) = 0 \quad \text{or} \quad (\lambda - 2h)^2 - 4(h^2 - a^2) = 0 \ldots \text{ (D)}.$$

Equation (D) will have real or imaginary roots for $\lambda$ according as $h^2$ is greater or less than $a^2$; that is, according as the distance of the centre of the circle defined by equation (A) is greater or less than the radius of the circle, or according as this circle and the radical axis meet in imaginary or real points. Hence the theorem follows.

The equations of the limiting points are $\left\{x + \tfrac{1}{2}(\lambda - 2h)\right\}^2 + y^2 = 0$, that is, $x = -\tfrac{1}{2}(\lambda - 2h)$, $y = 0$ (since each square must be separately zero, for a square of a real quantity cannot be negative), where $\lambda$ is to be found from equation (D).

Hence the coordinates of the limiting points are $\sqrt{h^2 - a^2}$, 0 and $-\sqrt{h^2 - y^2}$, 0.

### Exercise.

14. Find the limiting points of the system of coaxal circles $x^2 + y^2 + \lambda x - 5\lambda - 16 = 0$.

**34. To find the equation of the pair of tangents drawn from the point $(x_1, y_1)$ to the circle**

$$x^2+y^2+2gx+2fy+c=0.$$

Let $P$ be the point $(x_1, y_1)$, $PQ$ and $PR$ the tangents to the circle. Let $S$ denote the expression $x^2+y^2+2gx+2fy+c$, $S_1$ denote the expression $x_1^2+y_1^2+2gx_1+2fy_1+c_1$, $T$ denote the expression $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c$.

The equation $kS-T(lx+my+n)=0$ denotes a curve whose equation is of the second degree, and which passes through the points where $S=0$ is cut by $T=0$ and by $lx+my+n=0$.

Now $T=0$ is the equation of the chord of contact $QR$ of the tangents drawn from $P$; let $lx+my+n=0$ move up to and ultimately coincide with $T=0$.

Then the equation $kS-T^2=0$ denotes a curve whose equation is of the second degree, and which passes through a pair of coincident points on the circle at $Q$, and another pair of coincident points on the circle at $R$—that is, which is touched by the tangents $PQ$ and $PR$ at $Q$ and $R$.

Now the pair of straight lines $PQ$ and $PR$ form a curve answering to this description, and must therefore be a member of the system $kS-T^2=0$, found by ascribing some particular value to $k$. The value required is that value which will make the curve $kS-T^2=0$ pass through $(x_1, y_1)$; for the curve will then be cut by $PQ$ in *three* points (the point $P$ and two coincident points at $Q$) and by $PR$ also in three points, and this is impossible for a curve of the *second* degree unless the curve consists of the pair of straight lines $PQ$ and $PR$.

The value of $k$ required is therefore found by substituting the coordinates $(x_1, y_1)$ for $(x, y)$ in the equation $kS-T^2=0$. By this substitution $S$ becomes $S_1$, and $T$ becomes $S_1$, also.

$$\therefore \quad kS_1-S_1^2=0 \quad \text{or} \quad k=S_1$$

Hence the equation of the pair of tangents from $(x_1, y_1)$ to the circle is

$$\mathbf{S_1S-T^2=0} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \mathbf{(6)}.$$

The student must follow the reasoning of the above paragraph, which is of prime importance, with great care. The value of $k$ might have been deduced by applying the condition (found in Pt. I., § 32) that the equation

$$kS - T^2 = 0$$

should represent two straight lines, but the subsequent calculation would have been very cumbersome.

35. On account of the importance of the problem another solution will now be given. The coordinates of the point which divides the line joining $(x_1, y_1)$ to $(x_2, y_2)$ in the ratio $k : l$ are

$$(kx_2 + lx_1)/(k+l), \quad (ky_2 + ly_1)/(k+l) \quad \text{(Pt. I., § 3)}$$

By substituting the coordinates of this point in the equation of the circle we obtain a quadratic in $k/l$ giving the two ratios in which the line joining $(x_1, y_1)$ and $(x_2, y_2)$ is cut by the circle. If $(x_2, y_2)$ is on either tangent from $(x_1, y_1)$ to the circle, this quadratic must have equal roots.

The quadratic in question is

$$(kx_2 + lx_1)^2 + (ky_2 + ly_1)^2 + 2g(k+l)(kx_2 + lx_1)$$
$$+ 2f(k+l)(ky_2 + ly_1) + c(k+l)^2 = 0,$$

or $$k^2 S_2 + 2kl\, T_{12} + l^2 S_1 = 0 \quad \text{.................. (i.)},$$

where $$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c,$$
$$S_2 \equiv x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c,$$
$$T_{12} \equiv x_2(x_1 + g) + y_2(y_1 + g) + gx_1 + fy_1 + c.$$

The condition that equation (i.) should have equal roots is $S_1S_2 - T_{12}^2 = 0$. Hence, suppressing the accents of $x_2$ and $y_2$, the equation of the pair of tangents from $(x_1, y_1)$ to the circle is $$SS_1 - T^2 = 0 \quad \text{..................... (6)},$$ where $S$, $S_1$, and $T$ have the meanings given above.

### Exercises.

15. Show, by either of the above methods, and without assuming the formula established by §§ 31, 35, that the equation of the pair of tangents from $(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ is

$$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) - (xx_1 + yy_1 - a^2)^2 = 0.$$

16. Find the equation of the pair of tangents from the point (5, 5) to the circle $x^2 + y^2 - 6x - 6y + 14 = 0$.

36. **Expression of coordinates in terms of one parameter.**

The coordinates of any point on the circle $x^2+y^2=a^2$ may be written $(a\cos\alpha, a\sin\alpha)$ where $\alpha$ is the angle made with the axis of $x$ by the radius drawn to the point. the axes being assumed rectangular. This expression of the coordinates in terms of one unknown quantity $\alpha$ sometimes simplifies the form of the equations with which we have to deal. In particular, the equation of the tangent at the point $(a\cos\alpha, a\sin\alpha)$, to which we shall now refer as "the point $\alpha$," becomes

$$x\cos\alpha + y\sin\alpha = a.$$

**Exercises.**

17. Prove that the normal at the point $\alpha$ to the circle $x^2+y^2=a^2$ is

$$y = x\tan\alpha.$$

18. **Prove that the chord joining the points $\alpha$, $\beta$ is**

$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}.$$

19. Prove that the coordinates of the point of intersection of the tangents at $\alpha$ and $\beta$ are

$$\left(a\cos\frac{\alpha+\beta}{2}\sec\frac{\alpha-\beta}{2},\ a\sin\frac{\alpha+\beta}{2}\sec\frac{\alpha-\beta}{2}\right).$$

20. If the tangents at $Q$ and $R$ meet in $P$, and if $O$ be the centre and $OP$ cut $QR$ in $P'$, show that $OP \cdot OP' = OQ^2$.

37. **Inversion.** Definition.—Let $O$ be a given point and $P$ any point on a given curve. On $OP$ take a point $P'$ so that $OP \cdot OP' = k^2$, $k$ being a constant quantity. The locus of $P'$ as $P$ travels round the given curve is called the **inverse** of the given curve with regard to the centre $O$ and radius of inversion $k$.

**To find the inverse of a circle with regard to any point.**

Take the given point for origin and let the equation of the circle referred to polar coordinates be

$$r^2 - 2r\rho\cos(\theta-\alpha) + \rho^2 = a^2 \quad\ldots\ldots\ldots\ldots\ldots (1);$$

so that $(\rho, \alpha)$ are the polar coordinates of the centre and $a$ is the radius. If $(r', \theta)$ be the coordinates of the point on the inverse corresponding to $(r, \theta)$ on the circle, we have $rr' = k^2$ or $r = \frac{k^2}{r'}$.

Substituting in equation (1) and dropping the accent of $r'$, we see that the equation of the inverse is obtained by replacing $r$ by $\frac{k^2}{r}$ in equation (1), and is therefore

$$\left(\frac{k^2}{r}\right)^2 - 2\frac{k^2}{r}\rho\cos(\theta-\alpha) + \rho^2 = a^2$$

or $$r^2 - 2\frac{k^2\rho}{\rho^2-a^2} r\cos(\theta-\alpha) + \frac{k^4\rho^2}{(\rho^2-a^2)^2} = \frac{k^4a^2}{(\rho^2-a^2)^2} \quad \dots (2).$$

The inverse is therefore a circle whose centre is the point $\left(\frac{k^2\rho}{\rho^2-a^2}, \theta\right)$ and whose radius is $\frac{k^2a}{\rho^2-a^2}$.

To explain this equation geometrically, let $P'$ be the inverse of $P$, and let $OP$ cut the given circle again in $Q$. We have $OP \,.\, OP' = k^2$ and $OP \,.\, OQ =$ square of tangent from $O = \rho^2 - a^2$.

$$\therefore \quad OP' = \frac{k^2}{\rho^2-a^2} OQ;$$

so that the inverse is a reproduction of the original circle on the scale $\frac{k^2}{\rho^2-a^2}$, $P'$ being the reproduction of $Q$, and $Q'$ (the inverse of $Q$) being the reproduction of $P$.

If the centre of inversion $O$ is on the circle, the equation of the circle becomes $r = 2a\cos(\theta-\alpha)$ and that of the inverse $r\cos(\theta-\alpha) = \frac{k^2}{2a}$; so that the inverse is a straight line perpendicular to the radius through $O$. If the radius of inversion is $2a$, the inverse is the tangent at the point diametrically opposite to $O$.

**Exercise.**

21. If $P'$ is the inverse of $P$, show that the tangents to the curve at $P$ and to the inverse at $P'$ are equally inclined to $OPP'$.

### Illustrative Examples.

(i.) *Find the condition that the portion of the line $lx + my = 1$ intercepted by the circle $x^2 + 2xy\cos\omega + y^2 + 2gx + 2fy + c = 0$ shall subtend a right angle at the origin.*

If $P$ and $Q$ are the points of intersection, we must find the equation of $OP$, $OQ$, and put down the condition that the two lines are at right angles. Making the second equation homogeneous by means of the first, we find that $OP$, $OQ$ are given by

$$x^2 + y^2 + 2xy\cos\omega + 2(gx + fy)(lx + my) + c(lx + my)^2 = 0$$

or $x^2(1 + 2gl + cl^2) + 2xy\left\{\cos\omega + gm + fl + clm\right\} + y^2\left\{1 + 2fm + cm^2\right\} = 0.$

If these are at right angles, we have, by Pt. I, § 29,

$$(1 + 2gl + cl^2) + (1 + 2fm + cm^2) - 2(\cos\omega + gm + fl + clm)\cos\omega = 0,$$

or, on simplifying,

$$2\sin^2\omega + 2l(g - f\cos\omega) + 2m(f - g\cos\omega) + c(l^2 + m^2 - 2lm\cos\omega) = 0,$$

which is the required condition.

(ii.) *Prove that the difference between the squares of the tangents drawn from any point $K$ to two given circles is equal to twice the rectangle $KE \cdot AB$, where $KE$ is the perpendicular from $K$ on their radical axis, and $A$ and $B$ are their centres.*

Let the coordinates of $K$ be $x_1$, $y_1$.

Let the equations of the circles be $S' = 0$ and $S'' = 0$, where

$S' \equiv$
$x^2 + y^2 + 2g_1x + 2f_1y + c_1,$

$S'' \equiv$
$x^2 + y^2 + 2g_2x + 2f_2y + c_2,$

and let $S_1'$ and $S_1''$ denote the results of substituting the coordinates of $K$ in the expressions $S'$ and $S''$.

Let $KC$ and $KD$ be tangents from $K$ to the two circles.

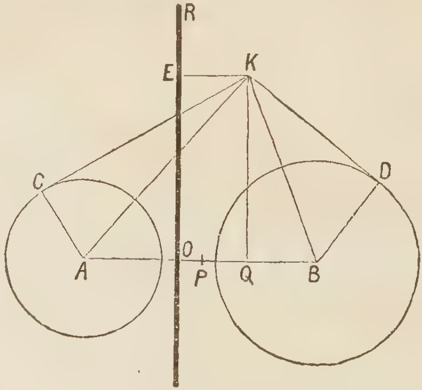


Fig. 10.

Then $KC^2 = S_1'$, $KD^2 = S_1''$.

$$\therefore \; KC^2 - KD^2 = S_1' - S_1'' \quad \ldots\ldots\ldots\ldots \text{(i.)}.$$

The equation of the radical axis is $S'-S''=0$, in which the coefficient of $x$ is $2(g_1-g_2)$ and that of $y$ is $2(f_1-f_2)$.

Hence $$KE=\frac{S_1'-S_2''}{2\sqrt{(g_1-g_2)^2+(f_1-f_2)^2}} \quad\text{(ii.)}.$$

(Pt. I., §17)

We are not concerned with the sign of $KF$. The coordinates of $A$ are $-g_1$, $-f_1$ and those of $B$ are $-g_2$, $-f_2$.

$$\therefore\quad AB=\sqrt{(g_1-g_2)^2+(f_1-f_2)^2} \quad\text{(3)}.$$

From (1), (2), and (3) we have $KC^2-KD^2=2KE\,.\,AB$.

(iii.) *$PT_1$, $PT_2$ are tangents drawn from a point $P$ to two given circles of a coaxal system. Prove that, as $P$ moves round the circumference of a third circle of the same system, the ratio $PT_1:PT_2$ is constant.*

With the same notation as in the last example, let $S'=0$, $S''=0$ be the equations of the two given circles. The equation of any other circle of the same system is of the form

$$S'+\lambda S''=0 \quad\text{(1)},$$

the geometrical interpretation of which is (§ 25)

$$PT_1^2+\lambda PT_2^2=0.$$

That is to say, as $P$ moves round the circumference of the circle defined by equation (1) $\dfrac{PT_1}{PT_2}=\sqrt{-\lambda}$, and so is constant. [Observe that $\lambda$ must be negative if $PT_1$ and $PT_2$ are to be real.]

(iv.) *Show that the tangent at the middle point of the arc cut off by a chord of a circle is parallel to the chord.*

Let $x^2+y^2=a^2$ be the equation of the circle.

Let $A$, $B$ be the extremities and $M$ the middle point of the arc.

With the notation of § 36, let $A$ be the point $\alpha$ and $B$ the point $\beta$; then $M$ is the point $\frac{1}{2}(\alpha+\beta)$.

The equation of the tangent at $M$ is

$$x\cos\tfrac{1}{2}(\alpha+\beta)+y\sin\tfrac{1}{2}(\alpha+\beta)=a \quad\text{(1)}.$$

That of the chord $AB$ is, by Ex. 18, p. 28,

$$x\cos\tfrac{1}{2}(\alpha+\beta)+y\sin\tfrac{1}{2}(\alpha+\beta)=a\cos\tfrac{1}{2}(\alpha-\beta) \quad\text{(2)}.$$

As the coefficients of $x$ and $y$ are the same in the two equations, it follows that the two lines are parallel.

(v.) *$ABCD$ is a quadrilateral inscribed in a circle. $AB$, $CD$ meet in $O$; $AC$, $BD$ in $P$, and $AD$, $BC$ in $Q$. Prove that $PQ$ is the polar of $O$.*

Take $OAB$, $OCD$ for axes of $x$ and $y$; let them include an angle $\omega$, and let the equation of the circle be

$$x^2+2xy\cos\omega+y^2+2gx+2fy+c=0 \quad\text{(1)}$$

Then $OA$ and $OB$ are the roots of the quadratic in $x$, $x^2+2gx+c=0$, and $OC$ and $OD$ are the roots of the quadratic in $y$, $y^2+2fy+c=0$.

Write $t^2$ for $c$, and let $OA = mt$, $OC = nt$.

Then $\qquad OB = t/m$

and $\qquad 2g = -t(m - 1/m)$;

also $\qquad OD = t/n$

and $\qquad 2f = -t(n + 1/n)$.

The equation of the circle therefore becomes

$$x^2 + 2xy\cos\omega + y^2 - (m + 1/m)\,tx - (n + 1/n)\,ty + t^2 = 0 \;\ldots\; (2),$$

and the polar of the origin $O$ with regard to (2) is, by Ex. 40, p. 33.

$$\tfrac{1}{2}(m + 1/m)\,x + \tfrac{1}{2}(n + 1/n)\,y - t = 0 \;\ldots\ldots\ldots\; (3).$$

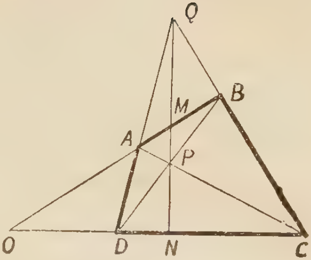



Fig. 11.

Now $PQ$ is a line through the intersection of $AC$ and $BD$, and also through the intersection of $AD$ and $BC$; hence (Pt. I., § 22) its equation must be of the forms

$$(x/m + y/n - t) + k(xm + yn - t) = 0$$

and

$$(x/m + yn - t) + k'(xm + y/n - t) = 0,$$

which can only be identical if $k = 1$, $k' = 1$; hence the equation of $PQ$ is

$$x(m + 1/m) + y(n + 1/n) - 2t = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (4),$$

that is, $PQ$ is the polar of $O$.

[Note that $t$ is the length of the tangent from $O$.]

## MISCELLANEOUS EXERCISES ON CHAP. II.

22. Tangents being drawn from the point $(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$, find the equations of the pair of straight lines joining their points of contact to the centre. If these lines are at right angles, show that the point $(x_1, y_1)$ lies on the circle $x^2 + y^2 = 2a^2$.

23. Show that the length of the tangent from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{c}$.

24. Find the radical axes of the following circles taken in pairs:—
$x^2 + y^2 - 3x - 6y + 8 = 0$, $x^2 + y^2 - x - 4y + 2 = 0$, $x^2 + y^2 + 2x - 6y + 3 = 0$; and find the point in which these three radical axes intersect.

25. Show that the length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + c' = 0$ is $\sqrt{c' - c}$.

26. Find the locus of the middle points of chords of the circle $x^2 + y^2 = a^2$ which pass through the fixed point $(h, k)$.

27. Prove that the locus of a point from which the tangents to two given circles are in a constant ratio is a circle passing through the points of intersection of the two given circles.

28. Find the coordinates of the radical centre of the three circles $x^2+y^2=9$, $x^2+y^2-2x-2y=5$, and $x^2+y^2+4x+6y=19$.

29. Prove that the circles $r=2a\cos(\theta-\alpha)$ and $r=2b\cos(\theta-\beta)$ cut at an angle $\alpha-\beta$.

30. Find the locus of the middle points of chords of a circle passing through a point on the circumference.

31. Find the equation of the circle through the origin and the points in which the circle $x^2+y^2+4x+3y+2=0$ is cut by the straight line $3x+2y+4=0$.

32. Find the equation of the tangents from the origin to the circle

$$(x-p)^2+(y-q)^2=r^2.$$

33. Show that the equations of any two circles can always be written as $x^2+y^2+py+c=0$, $x^2+y^2+qy+c=0$.

34. Find the condition that one of the circles in Ex. 33 lies entirely within the other.

35. Prove that the polar of the point $(p, q)$ with respect to $x^2+y^2=a^2$ touches $(x-c)^2+(y-d)^2=b^2$ if $b^2(p^2+q^2)=(a^2-cp-dq)^2$.

36. The pole of a straight line with regard to the circle $x^2+y^2=a^2$ lies on the straight line $ax+by=1$. Prove that the equation of the straight line is $x-ar^2=c(y-br^2)$, where $c$ is a constant.

37. Prove that the distances of two points from the centre of a circle are in the same ratio as the perpendicular distances of each point on the polar of the other.

38. Prove that the area of the triangle formed by the tangents from $(p, q)$ to the circle $x^2+y^2=r^2$ and the polar of $(p, q)$ is

$$\frac{r(p^2+q^2-r^2)^{\frac{3}{2}}}{p^2+q^2}.$$

39. Find the length of the common chord of the circles

$$(x+c)^2+(y+d)^2=r^2, \quad (x+d)^2+(y+c)^2=r^2.$$

40. Find the equation of a circle touching the axes of coordinates which are at right angles.

41. Find the equation to the common tangents to the circles

$$x^2+y^2-ay=0, \quad x^2+y^2-bx=0.$$

42. Find the equation of the circle coaxal with

$$x^2+y^2+2gx+2fy+c=0, \quad x^2+y^2+2Gx+2Fy+C=0$$

and passing through the origin.

43. The base and the ratio of the sides of a triangle are given. Prove that the locus of the vertex is a circle.

44. Find the locus of the points of contact of tangents, drawn in a given direction, to a system of coaxal circles.

45. Prove that, if a series of circles touch each other at one point, all their polars with respect to a given point are concurrent.

## EXAMINATION PAPER A.

1. A straight line of given length $(2a)$ is made to slide so that its extremities always lie on the axes of coordinates. Find the locus of its middle point, the axes being rectangular.

2. If the square of the distance of a point from the origin is $2a$ times its distance from the line $x = \frac{1}{2}a$, show that the locus is a point-circle, and find its position. The axes are supposed rectangular.

3. Explain what is meant by a "tangent to a curve."

4. The tangent at $P$ to the circle $x^2+y^2 = a^2$ meets the axes of $x$ and $y$ respectively in $T$ and $t$, and $PM$, $PN$ are drawn perpendicular on these axes. Prove that $CM \cdot CT = a^2$ and that $CN \cdot Ct = a^2$, $C$ being the centre of the circle.

5. Find the points of intersection of the circles $x^2+y^2 = 25$ and $x^2+y^2-26y+25 = 0$, and show that the circles cut at right angles.

6. Find the value of $p$ in order that the straight line

$$x \cos \alpha + y \sin \alpha = p$$

may touch the circle $x^2+y^2-2ax = 0$.

7. Show that the straight $y = mx$ touches the circles

$$x^2+y^2-2ax\sqrt{(1+m^2)}+a^2 = 0$$

and

$$x^2+y^2-2by\sqrt{(1+m^2)}+b^2m^2 = 0.$$

Deduce that the circles themselves touch if $a = \pm mb$.

8. Find the locus of the middle points of chords of a circle which pass through a fixed point (by taking this point for pole). Show that the locus is independent of the radius of the circle if the position of the centre remains unaltered.

9. Prove that, if the pole of a straight line with respect to the circle $x^2+y^2 = c^2$ lie on the circle $x^2+y^2 = 9c^2$, the polar is a tangent to the circle $x^2+y^2 = \dfrac{c^2}{9}$.

10. If from any point $O$ a line be drawn in any direction to meet a fixed straight line in the point $P$, and if a point $Q$ be taken in $OP$ so that the rectangle $OP \cdot OQ$ = a constant, then the locus of $Q$ is a circle.

# CHAPTER III.

## THE PARABOLA.

38. **Parabola.**—DEFINITIONS.—A **parabola** is the locus of a point which moves so that its distance from a fixed point is equal to its distance* from a fixed straight line.

[Of course we mean only in the plane where the point and line are. Throughout the book all points, lines, and curves considered are in one plane.]

The fixed point is called the **focus** and the straight line the **directrix.**

NOTE.—The focus is usually indicated by the letter $S$, and the foot of the perpendicular from $S$ on the directrix by the letter $X$.

Thus, if $P$ be a point on the curve and $PK$ the perpendicular from it on the directrix, we have $SP = PK$.

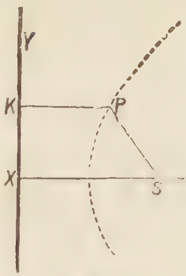



Fig. 12.

39. **Mechanical description of the parabola.**

From the definition given above we can deduce an easy method of describing a parabola.

Place a flat ruler with one edge along the directrix $XK$, and against the ruler place a triangular ruler $KDN$ so that one edge $KD$ is parallel to the axis of the curve. Fasten one end of a piece of thread, of length $DK$, to $D$, and the other

* The distance meant is, of course, the *shortest* distance, *i.e.*, it is the length of the *perpendicular* from the point to the given straight line.

end to the focus $S$. Then, if a pencil be placed against the thread and $DK$ so as to keep the thread tight, and the triangular ruler be moved along in contact with the fixed flat ruler, the pencil will describe a portion of a parabola. For $SP+PD$

$= \text{length of string}$
$= KD = KP+PD$;
$\therefore \quad SP = PK.$

The lower portion of the curve may be described by placing the triangular ruler, &c., in the dotted position.

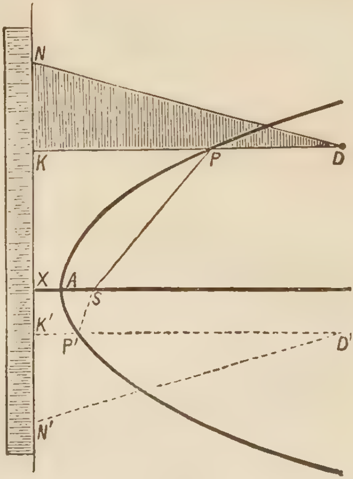


Fig. 13.

COR. **All parabolas are of the same shape, and differ only in size.**

From the definition and construction it is evident that any two parabolas are of the same *shape*, and that corresponding lines in them are proportional to their $SX$-distances, *i.e.* to the distances of foci from their respective directrices.

40. **To find the equation of a parabola.**

Let $S$ be the fixed point and $XK$ the fixed line, $SX$ being perpendicular to the directrix. Take $X$ for origin, $XS$ for axis of $x$, $XK$ for axis of $y$, and suppose that the known length $SX = 2a$. If $P\ (x, y)$ be any point on the curve, we have $SP = PK$;

$\therefore\ SP^2 = PK^2$, *i.e.* $(x-2a)^2+y^2 = x^2$,

or on reduction

$$y^2 = 4a\,(x-a),$$

which is the required equation.

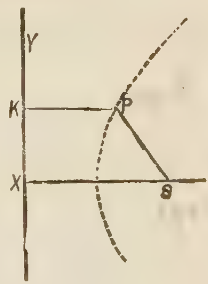


Fig. 14.

**41. Reduction of the equation of the parabola to the form** $y^2 = 4ax$.

If we take a new origin $A$ at the point $(a, 0)$, *i.e.* the middle point of $SX$, and axes parallel to the former ones, to find the new equation we have to replace $x$ by $x+a$ and $y$ by $y$ in the former equation (Part I., § 31).

Thus the equation becomes

$$y^2 = 4a(x+a-a) = 4ax,$$

or

$$y^2 = 4ax \qquad (1).$$

The equation $y^2 = 4ax$ is the simplest form to which the equation of a parabola can be reduced, and should be carefully remembered. The axes so chosen are called the *principal axes.*

**42. Shape of the parabola deduced from its equation.**

Taking $A$ for origin (Fig. 15), we have $SA = AX = a$, and the equation is $y^2 = 4ax$.

Thus
$$y = \pm 2\sqrt{ax},$$
so that every positive value of $x$ gives two equal and opposite values of $y$. Also, $a$ being supposed positive, when $x$ is negative $ax$ is negative, and therefore $y$ is unreal.

We immediately infer

(i.) the curve lies wholly to the right of the axis of $y$;

(ii.) any line parallel to $AY$ meets the curve in two points equidistant from the line $AX$;

(iii.) from (ii.) we infer that the line $AY$ or $x = 0$ is a tangent to the curve at $A$, for it gives two equal values of $y$, namely zero;

(iv.) as $x$ increases without limit, $y$ increases without limit.

These facts enable us to form a general idea as to the shape of the curve. It touches $AY$ at $A$, is symmetrical*

* A curve is said to be *symmetrical* with respect to a straight line when to each point of the curve on one side of the line there is a point on the *opposite* side on the perpendicular from the point to the line and at the same distance from it. If the straight line be regarded as a plane mirror, the portion of the curve on one side is the *image* of the portion on the opposite side. Hence every straight line perpendicular to the given straight line and terminated by the curve is bisected by the given straight line.

with respect to $XA$, and recedes indefinitely from the axis of $x$ on both sides as in the figure.

**Caution.**—When $a$ is negative, the curve lies wholly to the *left* of the axis of $y$, but (ii.), (iii.), (iv.) still hold good.

**Axis. Vertex.**—DEFINITIONS.—The straight line $XAS$, with regard to which the curve is symmetrical, is called the **axis** of the parabola, and the point $A$, in which the axis meets the curve, is called the **vertex.**

If $P$ be a point on the curve, and $PN$ the perpendicular on the axis, then $PN$ is called the **ordinate** of $P$, and, if it meets the curve again in $P'$, $PP'$ is said to be a double ordinate. The double ordinate $LSL'$ through the focus is called the **latus rectum.** Its length is $4a$ for, from

$$y^2 = 4ax,$$

we have

$$LS^2 = 4a \,.\, AS = 4a^2;$$

$$\therefore \quad LS = 2a.$$

Hence $LSL' = 4a$.

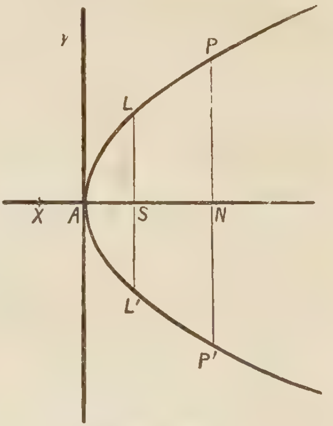



Fig. 15.

## Exercises.

1. **Show that the distance of the point $P\ (x',\ y')$ on the parabola $y^2 = 4ax$ from the focus is $a + x'$.**

2. What are the lengths of the latera recta of the following parabolas:—

$$\text{(i.) } y^2 = 2ax; \quad \text{(ii.) } y^2 = 7x; \quad \text{(iii.) } 7y^2 - x = 0\,?$$

3. The distance from the focus of a parabola to its vertex is 3. What is its equation in the simplest form?

4. $P$ is a point on the parabola $y^2 = 10x$ such that $AP$ makes an angle of (i.) 45°, (ii.) 30°, with the axis. Find the coordinates of $P$.
[In (i.) the ordinate of $P$ equals the abscissa.]

## 43. Plotting parabolas.

As explained in § 39, all parabolas are of the same shape, and differ only in size (and, of course, position). The student should become familiar with the shape of the curve by *plotting* out (on "squared" paper if possible) a few as in the following example.

*Example.—Plot out the curve* $y^2 = 3x$.

If we give $x$ the values

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,$$

the corresponding values of $y$ are

$$0,\ \pm\sqrt{3},\ \pm\sqrt{6},\ \pm\sqrt{9},\ \pm\sqrt{12},\ \pm\sqrt{15},\ \pm\sqrt{18},\ \pm\sqrt{21},\ \pm\sqrt{24}.$$

or $0,\ \pm 1{\cdot}7,\ \pm 2{\cdot}5,\ \pm 3,\ \pm 3{\cdot}5,\ \pm 3{\cdot}9,\ \pm 4{\cdot}2,\ \pm 4{\cdot}6,\ \pm 4{\cdot}9.$

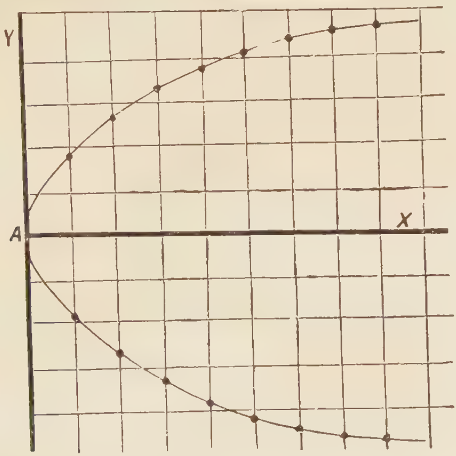



Fig. 16.

Hence, plotting these points (indicated by dots in the diagram) and drawing, roughly, a curve through them, we get a fair idea of the shape of the curve.

Note.—For the purposes of calculation, it would be easier to give the values $0,\ \pm 1,\ \pm 2, \ldots$ to $y$, and get, as corresponding values of $x$, $0,\ \frac{1}{3},\ \frac{4}{3},\ 3, \ldots$.

**Exercises.**

Plot out in the same way the following curves:—

5. (i.) $y^2 = 2x$; (ii.) $y^2 = 9x$.

6. (i.) $y^2 = -x$; (ii.) $y^2 = -7x$.

**44. Equation of parabola referred to axes parallel to its principal axes.**—Assuming that the reader is now familiar with the shape of the curve, we shall show how to trace a parabola when its equation is given referred to axes of coordinates *parallel* to the axis of the curve and tangent at its vertex.

*Example* (i.) *Trace the curve* $y^2-2x-2y+5=0$.

The equation must be rearranged so as to present the terms involving $y$ as a perfect square.

Thus $$y^2-2y=2x-5;$$

$$\therefore \quad (y-1)^2=2x-5+1=2(x-2).$$

Now transfer the origin to the point (2, 1) and the equation becomes

$$y^2=2x. \qquad \text{(Part I., § 31)}$$

[The point to which we transferred is therefore evidently the vertex of the curve.]

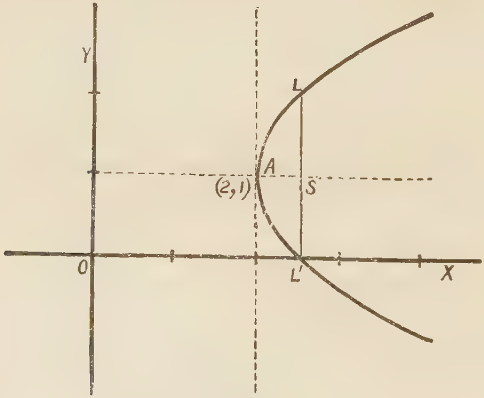



Fig. 17.

This is evidently a parabola with latus rectum = 2. It can easily be traced as in Fig. 17.

The student should confirm his work by noting where the curve cuts the *original* axes. Thus, when $y=0$, $x=\frac{5}{2}$, and, when $x=0$, $y^2-2y+5=0$, which gives imaginary values of $y$.

**Caution.**—The transferring of the origin does not alter the shape or size of the curve, but only alters its position relative to the axes of coordinates. It is clear that the position of the curve relative to the new axes of coordinates is not the same as its position relative to the old axes of coordinates.

*Example* (ii.) *Trace the curve* $x^2 = 4ay$.

Comparing this equation with $y^2 = 4ax$, we see that $x^2 = 4ay$ is the same as $y^2 = 4ax$ if we exchange the axes of $x$ and $y$, *i.e.* it is a parabola in which the axis of $x$ is the tangent at the vertex (and not the axis of the curve) and the axis of $y$ is the axis of the curve (and not the tangent at the vertex). It is therefore as in Fig. 18.

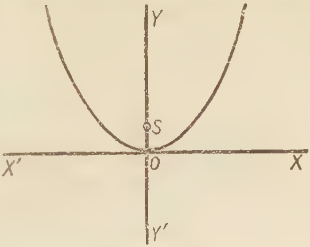



Fig. 18.

*Example* (iii.) *Trace the curve*

$$x^2-2x+2y-3 = 0.$$

As $x^2$ appears in the equation and not $y^2$, we must arrange the equation so as to present the terms involving $x$ as a perfect square.

Thus $\qquad (x-1)^2 = -2y+3+1 = -2(y-2).$

Transferring origin to point (1, 2), we get $x^2 = -2y$.

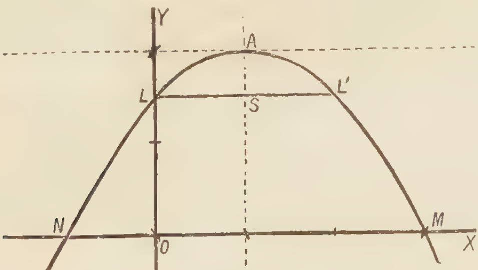



Fig. 19.

This is evidently a parabola with latus rectum = 2.

The *new* coordinate axis of $x$ is the tangent at the vertex, and the negative portion of the *new* coordinate axis of $y$ is the axis of the curve.

It cuts the *original* axis of $x$ where $y = 0$ and $x^2-2x-3 = 0$ or $x = 3$ or $-1$ ($M$, $N$), and the original axis of $y$ where $x = 0$ and $y = \frac{3}{2}$ ($L$). The curve is therefore as in Fig. 19.

## Exercises.

7. Trace the curve $\qquad y^2 = 2x-4.$

8. Show that the parabola $y^2 = -4ax$ is exactly equal to the curve $y^2 = 4ax$, but is turned in the opposite direction.
[Note that $x = -p$ gives the same ordinate in the one as $x = +p$ in the other.]

9. Trace the curve $y^2 = -3x+4$. Show that the length of its latus rectum is 3.

10. Trace the parabola $3y^2-x-18y+27=0$, and find the length of its latus rectum.

11. Trace the parabola $x^2-4x-8y-12=0$, and find the length of its latus rectum.

12. Where is the vertex of the curve $y^2=2x+3$? Find its focus and directrix.

13. Where is the vertex of the curve $(y+2)^2=x$? Find the equation of the tangent at the vertex.

**14. Show that a point $(x', y')$ is within, on, or outside the parabola $y^2=4ax$ according as $y'^2$ is less than, equal to, or greater than $4ax'$.**

15. Is the point (1, 2) within or outside the parabolas $y^2=2x$, $x^2=2y$? Draw a figure in illustration.

16. Show that the vertex of the curve $(y-p)^2=q(x-r)$ is at the point $(r, p)$, and that the latus rectum is $q$.

17. Find the vertices, foci, and directrices of the curves

$$\text{(i.)}\ (y-4)^2=2(x-1), \qquad \text{(ii.)}\ (y+3)^2=2(x+2).$$

**45. Whatever be the axes of reference, the equation of the parabola is of the second degree, and the terms of the second degree in $x$ and $y$ form a perfect square.**

In its simplest form the equation is $y^2=4ax$.

Now to transform to *any* other axes, rectangular or oblique, we replace $x$ and $y$ by *linear* functions of the new coordinates (Part I., § 35). Suppose we write $l_1x+m_1y+n_1$ for $x$, and $l_2x+m_2y+n_2$ for $y$; then the above equation becomes

$$(l_2x+m_2y+n_2)^2=4a(l_1x+m_1y+n_1),$$

which is of the second degree.

The terms of the second degree are $(l_2x+m_2y)^2$, and they form a perfect square.

46. If the new axes are rectangular, this may be proved in a different way. Let $(x', y')$ be the focus $S$, and $lx+my+n=0$ the equation of the directrix; then

$$SP^2=(x-x')^2+(y-y')^2,$$

and

$$PK^2=\frac{(lx+my+n)^2}{l^2+m^2}; \qquad \text{(Part I., § 17)}$$

$$\therefore \quad (x-x')^2+(y-y')^2=\frac{(lx+my+n)^2}{l^2+m^2},$$

which is of the second degree in $x$ and $y$.

Cleared of fractions it becomes

$$(l^2+m^2)\left\{(x-x')^2+(y-y')^2\right\}-(lx+my+n)^2=0.$$

Now the terms in the second degree in $x$ and $y$ are

$$(l^2+m^2)(x^2+y^2)-(lx+my)^2,$$

and this equals $$(ly-mx)^2;$$

therefore they form a perfect square.

Thus, if an equation of the second degree such as

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a parabola, the terms

$$ax^2+2hxy+by^2$$

must be a perfect square in $x$ and $y$.

The condition for this is $ab=h^2$.

We shall see in § 52 that the converse of this proposition is also true, viz., that

*If the terms of the second degree form a perfect square, then the curve is a parabola.*

*Example.—Find the equation and the length of the latus rectum of the parabola whose focus is at the point* (1, 1) *and whose directrix is the line*

$$3x+4y=1.$$

If $(x, y)$ be any point on the curve, its distance from (1, 1) must be equal to its distance from the line $3x+4y=1$.

Hence $$\sqrt{(x-1)^2+(y-1)^2}=\frac{3x+4y-1}{\sqrt{3^2+4^2}};$$ leading to

$$25x^2-50x+25+25y^2-50y+25=9x^2+24xy+16y^2-6x-8y+1.$$

$$\therefore\quad 16x^2-24xy+9y^2-44x-42y+49=0.$$

The latus rectum is twice the perpendicular from the focus on the directrix, *i.e.* it is $$2\,\frac{3.1+4.1-1}{\sqrt{25}}=\tfrac{12}{5}.$$

**Exercises.**

18. Find the equation of the parabola whose focus is $(a, 0)$ and directrix $x+a=0$.

19. Find the equation of the parabola whose focus is (1, 2) and directrix $x+y-2=0$.

20. Find the latera recta of the parabolas in Ex. 18 and 19.

**47. A straight line meets a parabola in two points.**

Let $y = mx + c$ be the equation of the straight line, and $y^2 = 4ax$ the equation of the parabola; then to find the points of intersection we have to solve these two equations for $x$ and $y$.

Substituting for $y$ its value in terms of $x$, we find

$$(mx + c)^2 = 4ax \text{ .................... (A)},$$

which is a quadratic equation for $x$ and has therefore two roots. If $x_1$ and $x_2$ be these roots, then these are the abscissæ of the common points, and their ordinates are given by $y_1 = mx_1 + c$, $y_2 = mx_2 + c$. Thus there are two points of intersection, viz., $(x_1, y_1)$ and $(x_2, y_2)$.

If $m = 0$, one of the roots of the quadratic is infinite. There are still two points, but one is at infinity.

NOTE.—When the roots of (A) are imaginary, the straight line meets the curve in two imaginary points.

This proposition may be illustrated as follows:—Let $P$ be a point on $OY$, and $PQR$ be a straight line through $P$. Suppose this line moves from the position $PY'$ in the counter-clockwise direction to the position $PY$, $m$ thus changing from $-\infty$ through zero to $+\infty$. Then at first it will cut the parabola in two points $Q_1$, $R_1$, and equation (A) will have real roots for that value of $m$ corresponding to the position $PQ_1R_1$. When the line is in position $PQ_2$ parallel to $OX$, $m = 0$ and one root is infinite. After this both roots are real and finite, as in $PQ_3R_3$, until the line touches the curve as in position $PQ_4$; then the roots are equal. After this both roots become imaginary, and the line does not meet the curve in real points until it comes again into the position $PY$.

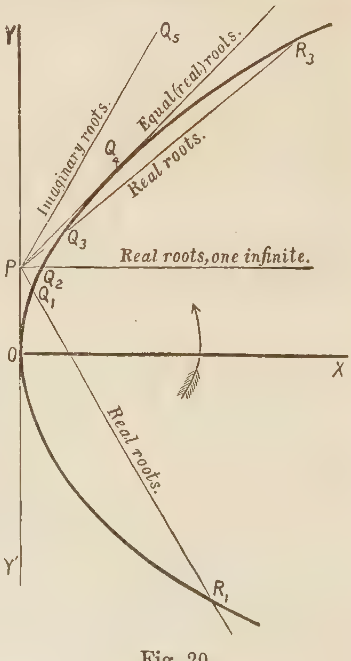



Fig. 20.

**48. To find the condition that the line $y = mx + c$ should touch the parabola $y^2 = 4ax$.**

If the line $y = mx + c$ touches the curve, the two points of intersection coincide, and thus the quadratic in $x$ must have equal roots. Now it is

$$(mx + c)^2 - 4ax = 0$$

or
$$x^2 . m^2 + 2x(cm - 2a) + c^2 = 0,$$

and the condition for equal roots is therefore

$$(cm - 2a)^2 = c^2m^2$$

or
$$-4acm + 4a^2 = 0;$$

$$\therefore \quad cm = a \quad \text{or} \quad c = a/m.$$

Thus the line $\qquad y = mx + a/m$ .................... (2)

touches the parabola for all values of $m$.

**Caution.**—When $m = \infty$, *i.e.*, when the line is parallel to the axis of $y$, the proof fails, for the equation is of the form $x = c$, and we cannot eliminate $y$ by means of it. In this case we get *one* value of $x$ to two equal and opposite values of $y$, but the line $x = c$ is *not* a tangent, except when $c = 0$.

49. The parabola not being a closed curve, some lines may meet it in points which are at an infinite distance from the origin.

In this case the quadratic for $x$ must have one or more infinite roots. Now it may be written

$$x^2 . m^2 + 2x(mc - 2a) + c^2 = 0,$$

and this equation has an infinite root if

$$m^2 = 0,$$

*i.e.*, if
$$m = 0. \qquad (\textit{Tut. Alg.}, \text{ II., § 166})$$

Thus *any line parallel to the axis of a parabola meets the curve in one point at an infinite distance.*

For both roots of the quadratic to be infinite we must have $\qquad m^2 = 0 \quad \text{and} \quad mc - 2a = 0,$

leading either to $a = 0$, which is contrary to hypothesis, or to $c = \infty$; and, if the latter be true, the line is at infinity.

Hence no line at a finite distance meets the curve in two points at an infinite distance.

50. **Any equation of the form $y^2 = ax + by$ denotes a parabola, the axes being rectangular.**

This equation can be written

$$y^2 - by = ax,$$

or, completing the square with respect to $y$,

$$\left(y - \frac{b}{2}\right)^2 = ax + \frac{b^2}{4} = a\left(x + \frac{b^2}{4a}\right),$$

and now, taking the point $\left(-\frac{b^2}{4a}, \; +\frac{b}{2}\right)$ for new origin, the equation becomes

$$Y^2 = aX, \qquad \text{(Part I., § 31)}$$

which is an equation of the same kind as that found in § 41, and therefore denotes a parabola.

51. **If a point moves so that the square of its perpendicular on one line varies as its perpendicular on another line, it describes a parabola.**

Take the point of intersection for origin and the first line for axis of $x$. Then the equation of the second is of the form

$$y - mx = 0.$$

Hence the equation is

$$y^2 = k\,\frac{y - mx}{\sqrt{1 + m^2}}, \qquad \text{(Part I., § 17)}$$

where $k$ is a constant. But this equation is plainly of the same form as

$$y^2 = px + qy,$$

and therefore it denotes a parabola.

**52. If in the equation**

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

**the terms of the second degree ($i.e.$, $ax^2 + 2hxy + by^2$) form a perfect square, the equation denotes a parabola.**

For suppose $ax^2 + 2hxy + by^2 = (\alpha x + \beta y)^2$;

then we have $(\alpha x + \beta y)^2 = -(2gx + 2fy + c)$.

Now the square of the perpendicular from $(x, y)$ on the line $\alpha x + \beta y = 0$ is

$$\frac{(\alpha x + \beta y)^2}{\alpha^2 + \beta^2},$$

and the perpendicular on $2gx + 2fy + c = 0$ is

$$\frac{2gx + 2fy + c}{2\sqrt{g^2 + f^2}}.$$

Hence for points on the curve the ratio of the first to the second

$$= \frac{(\alpha x+\beta y)^2}{\alpha^2+\beta^2} \Big/ \frac{2gx+2fy+c}{2\sqrt{g^2+f^2}}$$

$$= \frac{2\sqrt{g^2+f^2}}{\alpha^2+\beta^2},$$

which is constant.

Consequently, by § 51, the curve represented by the equation is a parabola.

**Exercises.**

21. Show that the equation $y^2 = x + y$ denotes a parabola, and find the coordinates of its vertex.

22. Show that the equation $y^2 = 2x + 4y + 1$ denotes a parabola, and find the coordinates of its vertex; also the equations of the axis and the tangent at the vertex.

[By changing the origin, we can reduce the equation to the form $Y^2 = 2X$. Now $X = 0$ and $Y = 0$ are the tangent at the vertex and the axis, so we have only to change back to the old origin.]

23. Show that any equation of the form $x^2 = ax + by$ denotes a parabola having its axis parallel to $OY$.

24. Show that the vertex of the parabola in Ex. 23 is at the point

$$\left(\frac{a}{2}, -\frac{a^2}{4b}\right).$$

25. Find the vertex, focus, and directrix of the parabola $x^2 = 2x + y$.

[First find the vertex and the length of the latus rectum; then use a figure.]

26. Show that the axis of the parabola $(\alpha x + \beta y)^2 + 2gx + 2fy + 1 = 0$ is parallel to the straight line $\alpha x + \beta y = 0$.

## MISCELLANEOUS EXERCISES ON CHAP. III.

27. Find the equation of the parabola whose focus is at $(-1, 1)$ and whose directrix is $x + y + 2 = 0$.

28. Find the latus rectum of the parabola in Ex. 27.

29. Find the coordinates of the point of intersection of the line $x + y = 3$ with the parabola $y^2 = 5x$.

30. Find the condition that the line $y = mx + c$ should touch the parabola $y^2 = 4a(x + b)$.

31. Show that the line $y = mx + c$ will touch the parabola

$$y^2 = Ax + By + C$$

if

$$c = \frac{1}{4mA}(A + Bm)^2 + \frac{Cm}{A}.$$

[Eliminate $x$ and find a quadratic for $y$.]

32. Find the coordinates of the point of intersection of the straight line $3x - y + 2 = 0$ with the parabola $y^2 = 2x + 1$.

33. If $P$, $Q$ be the points of intersection in Ex. 29, find the coordinates of the middle point $R$ of $PQ$.

[Note that, if $R$ be $(\xi, \eta)$, $P$ $(x_1, y_1)$, $Q$ $(x_2, y_2)$, then $\xi = \frac{1}{2}(x_1 + x_2)$, and use the theory of quadratics.]

34. Find the length of the side of an equilateral triangle inscribed in the parabola $y^2 = 4ax$ so that one angular point is at the vertex.

35. Find the length of the side of an equilateral triangle inscribed in the parabola $y^2 = 4ax$ so that one angular point is at the focus.

36. A straight line through the focus of a parabola meets the curve in $P$ and $P'$. Show that the product of the ordinates of $P$ and $P'$ is equal to the square on half the latus rectum.

37. In the same case, the product of the abscissæ is always equal to the square on one-fourth of the latus rectum.

38. Show that any line parallel to $x + y = 0$ will meet the parabola $x^2 + 2xy + y^2 + 2x + 4y + 1 = 0$ in one point at infinity. Hence $x + y = 0$ is parallel to the axis.

39. The axis of a parabola is $x \cos\alpha + y \sin\alpha - p = 0$, the tangent at the vertex is $x \sin\alpha - y \cos\alpha = 0$, and the latus rectum is $4a$. Show that the equation is

$$(x \cos\alpha + y \sin\alpha - p)^2 = \pm 4a\,(x \sin\alpha - y \cos\alpha)$$

according to which side of the tangent at the vertex the parabola is.

[Use the result $PN^2 = 4AS.AN$ where $PN$ is the perpendicular on the axis and $AN$ is equal to the perpendicular on the tangent at the vertex.]

40. Find the equation of the parabola whose axis is $x + y - 2 = 0$, tangent at the vertex $x - y = 0$, and latus rectum $\sqrt{2}$, the parabola lying on the same side of the tangent at the vertex as the point $(2, 1)$ does.

41. Find the equation of a parabola having the same axis, vertex, and latus rectum as the foregoing, but lying on the other side of the tangent at the vertex.

42. Prove that, if the straight line $\lambda x + \mu y + \nu = 0$ touches the parabola $y^2 - 4px + 4pq = 0$, then must

$$\lambda^2 q + \lambda\nu - p\mu^2 = 0$$

## EXAMINATION PAPER I.

1. Show how to find the joint equation of the lines joining the origin to the points of intersection of

$$ax^2 + by^2 + 2gx + 2fy = 0 \quad \text{and} \quad y = mx + c,$$

and hence determine the condition that $y - c$ is a tangent to

$$ax^2 + by^2 + 2gx + 2fy = 0.$$

2. $ABC$ is a triangle. $DE$ is a variable line, parallel to $BC$, and cutting $AB$ in $D$ and $AC$ in $E$. Find the locus of the point of intersection of $BE$ and $CD$.

3. Through what angle must the axes, supposed rectangular, be turned to transform $ax^2 + 2hxy + by^2$ into $a'x^2 + b'y^2$. Deduce a simple relation between $a'$, $b'$, $a$, $b$.

4. Prove that the degree of an equation is unaltered by any change of axes.

5. As in Example, p. 39, plot out (i.) the parabola $x^2 = 3y + 7$, (ii.) the curve $100y = 4x^3 + x$.

6. Draw the curve $x^2 - 2ax + 3ay + a^2 = 0$, finding (i.) the equation of the directrix, (ii.) the coordinates of the extremities of the latus rectum, and confirm your work by finding the intercepts on the axis of $y$.

7. Find the equation and length of the latus rectum of the parabola whose focus is $(-a, 0)$, directrix $x + y + 2a = 0$.

8. Show that the straight line $x + ny + an^2 = 0$ touches the parabola $y^2 = 4ax$, and find the point of contact.

9. Show that the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents a parabola if $h^2 = ab$.

10. Prove that the area of a triangle inscribed in the parabola

$$y^2 = 4ax \qquad = \frac{1}{8a}(l \sim m)(m \sim n)(n \sim l)$$

where $l$, $m$, $n$ are the ordinates of the angular points.

# CHAPTER IV.

## THE ELLIPSE.

53. **Ellipse.**—DEFINITIONS.—An **ellipse** is the locus of a point which moves so that its distance from a fixed point bears a constant ratio of lesser inequality to its distance* from a fixed straight line.

The fixed point is called a **focus** and the fixed line the corresponding **directrix**, and the constant ratio the **eccentricity**.

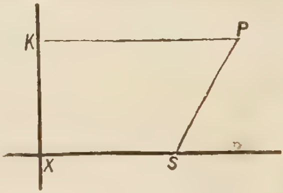



Fig. 21.

The eccentricity is usually denoted by the letter $e$.

For an ellipse, therefore, $e$ is less than 1.

Thus, if $P$ be a point on the curve, and $PK$ the perpendicular from it on the directrix, we have

$$SP = e.PK \quad \ldots\ldots\ldots\ldots \quad (1).$$

54. **To find the equation of an ellipse.**

Let $S$ be the focus and $XK$ the directrix, $SX$ being perpendicular to it (Fig. 21).

Take $X$ for origin, and $XS$, $XK$ for axes.

Let $e$ be the eccentricity, and put $SX = p$.

Then, if the point $P$ (whose coordinates are $x$, $y$) is on the curve, we have $\quad SP = e.PK$;

$$\therefore \quad SP^2 = e^2.PK^2,$$

i.e., $$(x-p)^2 + y^2 = e^2x^2;$$

$$\therefore \quad x^2(1-e^2) + y^2 - 2px + p^2 = 0,$$

which is the required equation of the ellipse.

* See footnote, p. 35.

**55. Reduction of the equation of the ellipse to the form** $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$

By changing the origin, we can reduce the equation obtained in § 54 to a much simpler form, which is the one generally used. The equation may be written

$$(1-e^2)\left\{x^2-2x\frac{p}{1-e^2}\right\}+y^2+p^2=0.$$

Consequently, completing the square with respect to $x$, we have $(1-e^2)\left\{x-\dfrac{p}{1-e^2}\right\}^2+y^2+p^2=\dfrac{p^2}{1-e^2}.$

Now let us take as new origin a point on $SX$ distant $p/(1-e^2)$ from $X$, and axes parallel to the original axes; then, referred to the new axes, the equation is, by Part I., § 31

$$(1-e^2)\,x^2+y^2=\frac{p^2}{1-e^2}-p^2=\frac{p^2e^2}{1-e^2}\text{ or }x^2+\frac{y^2}{1-e^2}=\frac{p^2e^2}{(1-e^2)^2}.$$

Now put $$p^2e^2/(1-e^2)^2=a^2.$$

Then the above becomes, on dividing across by $a^2$,

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1;$$

and, putting $$a^2(1-e^2)=b^2,$$

this is $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\quad (2).$$

Cor. $$e^2=1-\frac{b^2}{a^2}.$$

*Example.—Find the equation of the ellipse having* (1, 1) *for focus,* $2x-y+1=0$ *for corresponding directrix, and eccentricity* $1/\sqrt{2}$.

Here, if $P$ be $(x, y)$, we have $SP^2=(x-1)^2+(y-1)^2$; $PK$ = perpendicular from $P$ on directrix $=(2x-y+1)/\sqrt{5}$.

But $$SP^2=e^2.PK^2=\tfrac{1}{2}PK^2.$$

Hence $$(x-1)^2+(y-1)^2=\tfrac{1}{10}(2x-y+1)^2$$

is the required equation. Simplifying and transposing, we get

$$6x^2+4xy+9y^2-24x-18y+19=0.$$

**Exercises.**

1. Find the equation of an ellipse having (0, 1) for focus, $x+y=0$ for directrix, and eccentricity $\frac{1}{2}$.

2. Find the equation of an ellipse having $(\sqrt{a^2-b^2}, 0)$ for focus, $x=a^2/\sqrt{a^2-b^2}$ for directrix, and eccentricity $\sqrt{a^2-b^2}/a$.

**56. Shape of the ellipse.**

The shape of the ellipse is easily found from the equation

in its simple form, $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

We have $$y^2=b^2\left(1-\frac{x^2}{a^2}\right)$$

and, since $y^2$ must be positive, we see that

$$1 \nless \frac{x^2}{a^2},$$

or $x$ cannot be *numerically* greater than $a$.

Similarly, $y$ cannot be greater than $b$.

We have, from the above,

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}.$$

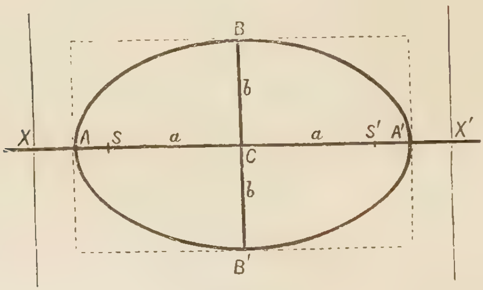



Fig. 22.

Hence (i.) $x$ cannot be numerically greater than $a$.

(ii.) $x=\pm a$ makes $y=0$. Thus, since the lines $x=\pm a$ make the two values of $y$ equal, they are both tangents.

(iii.) A value of $x$ numerically less than $a$ gives two equal and opposite values of $y$.

Thus the curve lies entirely between the lines $x=+a$ and $x=-a$, and every chord parallel to the axis of $y$ is bisected by the axis of $x$, *i.e.*, the curve is symmetrical* with respect to the axis of $x$.

* See footnote, p. 37.

Similarly, from the equation

$$x = \pm \frac{a}{b} \sqrt{b^2 - y^2},$$

we infer that the curve lies entirely between the lines $y = +b$ and $y = -b$, and is symmetrical with respect to the axis of $y$.

Further, as one coordinate increases numerically, the other decreases numerically; hence the curve is an oval of the form shown.

If $A$, $A'$ be taken on the axis of $x$ such that

$$CA = CA' = a,$$

$C$ being the origin, and $B$, $B'$ on the axis of $y$ such that

$$CB = CB' = b,$$

then $AA'$ and $BB'$ are called the **major and minor axes** of the ellipse, and the point $C$ is called the **centre.**

Cor. The curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, when $b > a$, is evidently an ellipse whose major axis $b$ lies along the axis of $y$, and minor axis $a$ along axis of $x$. Its foci* are, therefore, on the axis of $y$, its directrices* are parallel to the axis of $x$, and its eccentricity is given by $a^2 = b^2(1 - e^2)$.

### Exercises.

3. Sketch roughly, in one diagram, the curves
(i.) $x^2/9 + y^2/4 = 1$; (ii.) $x^2 + 4y^2 = 1$; (iii.) $4x^2 + y^2 = 1$.

4. Find the length of the ordinates of each of the curves in Ex. 3 corresponding to the middle points of the semi-major axis ($CA$).

5. **Show that a point $(h, k)$ is outside, on, or within the curve $x^2/a^2 + y^2/b^2 = 1$, according as**

$$h^2/a^2 + k^2/b^2 >, =, < 1.$$

6. Determine whether the point $\left(\dfrac{2}{\sqrt{10}}, \dfrac{3}{\sqrt{10}}\right)$ is outside or within the curves (i.) $x^2/4 + y^2 = 1$, (ii.) $x^2 + y^2/4 = 1$; and illustrate your answer with a diagram.

---

* See enunciation of proposition in § 60.

**57. Equation of ellipse referred to axes parallel to the principal axes.**—In the reduced equation

$$x^2/a^2 + y^2/b^2 = 1$$

the centre of the ellipse is the origin of coordinates and the axes of the curve are the axes of coordinates.

Frequently, however, the equation of the curve is given referred to axes of coordinates parallel to, but not both coincident with, the axes of the curve. In such a case there is no difficulty in tracing the curve. In the case of the parabola, we managed this by transferring the origin to the vertex of the parabola. In the case of the ellipse (and of the hyperbola, as in Chap. V.), we transfer the origin of coordinates to the "centre" of the curve.* The method of procedure will be seen in the following examples:—

*Example* (i.). *Trace the curve*

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1.$$

If we transfer origin to the point $C$ (1, 2), the equation at once becomes $\frac{x^2}{9} + \frac{y^2}{4} = 1$. (Part I., § 31.)

This is clearly an ellipse with axes 3 and 2 along the axes of coordinates.

Confirm the diagram by finding intercepts on original axes.

$x = 0$ gives

$$\frac{(y-2)^2}{4} = 1 - \tfrac{1}{9} = \tfrac{8}{9},$$

or $y = 2 \pm \frac{4}{3}\sqrt{2} = 3{\cdot}88$ or $\cdot 12$.

$y = 0$ gives

$$\frac{(x-1)^2}{9} = 1 - 1, \quad \text{or} \quad x = 1.$$

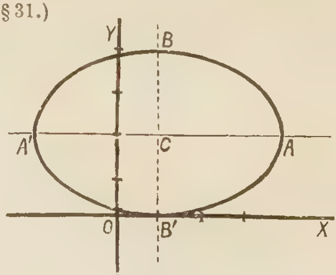



Fig. 23.

*Example* (ii.). *Trace the curve* $x^2 + 4y^2 - 2x - 16y + 8 = 0$.

Here arrange the terms in $x^2$ and $x$, together with some numerical quantity, so as to form a perfect square, and do similarly for the terms in $y^2$ and $y$.

Thus $(x^2 - 2x + 1) + 4(y^2 - 4y + 4) - 9 = 0,$

or $(x-1)^2 + 4(y-2)^2 = 9,$

or $$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{\frac{9}{4}} = 1.$$

---

* See Caution on p. 40.

Transferring origin to (1, 2), we get the equation

$$\frac{x^2}{9}+\frac{y^2}{\frac{9}{4}}=1,$$

which is evidently an ellipse with axes 3 and $\frac{3}{2}$, and is as in figure. It cuts the original axes $\quad x=0$

where $4y^2-16y+8=0$,

or $\quad y=2\pm\sqrt{2}=3{\cdot}4$ or $\cdot 6$;

and $\quad y=0$

where $\quad x^2-2x+8=0$,

*i.e.* in imaginary points.

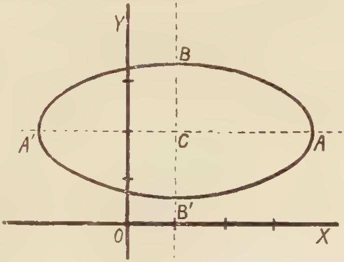



Fig. 24.

*Example* (iii.). *Trace the curve* $2x^2+y^2-2y-3=0$.

As in Example (ii.), we collect terms thus:—

$$2x^2+(y^2-2y+1)-4=0,$$

or $$2x^2+(y-1)^2=4,$$

or $$\frac{x^2}{2}+\frac{(y-1)^2}{4}=1.$$

On transferring origin to the point (0,1), we get the equation

$$\frac{x^2}{2}+\frac{y^2}{4}=1.$$

This is an ellipse with axes $\sqrt{2}$ and 2, but evidently the major axis is along the (new) axis of $y$. The curve is, therefore, as in figure.

Intercepts on original axes are given by $\quad x=0$,

$y^2-2y-3=0$ or $y=3$ or $-1$,

and $y=0$, $x=\pm\sqrt{\frac{3}{2}}=\pm 1{\cdot}2$.

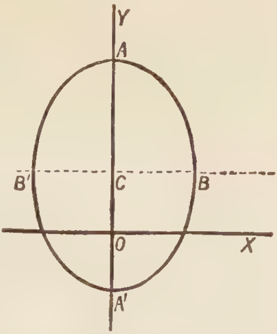



Fig. 25.

*Example* (iv.) *Find* (*a*) *the eccentricity*, (*b*) *the coordinates of the ends of the major axis*, (*c*) *the coordinates of the ends of the minor axis, of the curve in Example* (ii.), *all referred to the original axes.*

(*a*) In that curve, $\quad a=3 \quad$ and $\quad b=\frac{3}{2}$.

$\therefore \quad e^2=1-b^2/a^2=1-(\frac{3}{2}/3)^2=1-\frac{1}{4}=\frac{3}{4}; \quad \therefore \quad e=\sqrt{3}/2.$

(*b*) The coordinates of $C$ are 1, 2, and $ACA'$ is parallel to axis of $x$. Therefore, since $CA=CA'=3$, the coordinates of $A$ are $(\overline{1+3}, 2)$ or (4, 2), and those of $A'$ are $(\overline{1-3}, 2)$ or $(-2, 2)$.

(*c*) Since $CB=CB'=\frac{3}{2}$ and $BCB'$ is parallel to axis of $y$, the coordinates of $B$ and $B'$ are, respectively, $(1, \overline{2+\frac{3}{2}})$, $(1, \overline{2-\frac{3}{2}})$, or $(1, \frac{7}{2})$, $(1, \frac{1}{2})$.

**Exercises.**

Trace the following curves:—

7. $(x-2)^2/4+(y-1)^2=1$.
8. $(x-1)^2/9+(y-2)^2=4$.
9. $x^2+4y^2-8y=0$.
10. $4x^2+y^2-2y-3=0$.
11. $x^2+2y^2+2x-4y-1=0$.
12. $15x^2+4y^2-90x+32y-41=0$.

13–18. In each of the above curves (7–12), find

(i.) the eccentricity;

(ii.) the coordinates of the ends of the major axis;

(iii.) the coordinates of the ends of the minor axis.

58. **Polar equation of the ellipse, the centre being the pole.**

To find the polar equation, we must (by Part 1, § 6) put
$$x=r\cos\theta \quad \text{and} \quad y=r\sin\theta$$
in the equation
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Hence we get
$$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)=1,$$
or
$$\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2} \quad \text{.......  ........} \quad \textbf{(3),}$$
as the required polar equation.

59. **Shape of the curve derived from the polar equation.**

Equation (3) can be written
$$\frac{1}{r^2}=\frac{1}{a^2}+\sin^2\theta\left(\frac{1}{b^2}-\frac{1}{a^2}\right).$$

Now, as $a>b$, $\qquad \dfrac{1}{b}>\dfrac{1}{a}$.

Therefore, as $\theta$ increases from 0 to $\dfrac{\pi}{2}$, the expression on the right increases.

Therefore $\dfrac{1}{r}$ increases, *i.e.*, $r$ diminishes, *i.e.*, as $\theta$ increases from 0 to $\frac{1}{2}\pi$, $r$ steadily diminishes.

The same happens in each quadrant, viz., in passing

from an end of the major to an end of the minor axis, the radius vector constantly diminishes.

This affords a good method of tracing the curve, since we can easily find the distance from the centre in which any line meets the curve.

*Example* (i.). *In the ellipse whose semi-axes are* 2 *and* 1, *respectively, find the length of the radius vector from the centre making an angle* 45° *with the major axis.*

The Cartesian equation of the curve referred to its major and minor axes as axes of coordinates is

$$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1.$$

Hence the polar equation, with centre as pole, gives

$$\frac{1}{r^2} = \frac{\cos^2\theta}{4} + \frac{\sin^2\theta}{1} = \frac{\frac{1}{2}}{4} + \frac{\frac{1}{2}}{1} = \tfrac{5}{8}.$$

Hence
$$r = \sqrt{\tfrac{8}{5}}.$$

*Example* (ii.). **In any ellipse the sum of the reciprocals of the squares of two radii vectores at right angles is constant.**

Suppose they are $r$, making an angle $\theta$ with $CA$, and $r_1$, making an angle $\left(\theta + \frac{\pi}{2}\right)$ with $CA$. Then

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2},$$

$$\frac{1}{r_1^2} = \frac{\cos^2\left(\theta + \frac{\pi}{2}\right)}{a^2} + \frac{\sin^2\left(\theta + \frac{\pi}{2}\right)}{b^2} = \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}.$$

Hence, adding, we have

$$\frac{1}{r^2} + \frac{1}{r_1^2} = \frac{\cos^2\theta + \sin^2\theta}{a^2} + \frac{\cos^2\theta + \sin^2\theta}{b^2}$$
$$= \frac{1}{a^2} + \frac{1}{b^2},$$

which proves the result required.

**Exercises.**

19. Find, in the curves of Ex. 3, p. 53, the lengths of the radii vectores from the centre, making (i.) 45°, (ii.) 60°, with the major axis.

20. Sketch, in one diagram, the curves $x^2/4 + y^2 = 1$, $x^2/9 + 4y^2 = 1$; find the tangent of the angle the common radius vector makes with the axis of $x$, and find the length of the radius vector.

60. **To show that the ellipse has a second focus and a second directrix.**

Since the curve is quite symmetrical with respect to $ACA'$ and $BCB'$, if we take $S'$ and $X'$ on $ACA'$ so that $CS' = CS$ and $CX' = CX$, and draw $X'K'$ perpendicular to $CX'$, then it follows that $S'$ is a second focus and $X'K'$ another directrix, and the *whole* of the ellipse can clearly be generated from them in exactly the same way as it can from the original focus and directrix.

61. **To show that $CX = a/e$ and $CS = ae$.**

(i.) Since $A$ and $A'$ are on the curve, we have

$$\left.\begin{aligned} SA &= e\,.\,AX \\ SA' &= e\,.\,A'X \end{aligned}\right\} \text{(Fig. 26)} \;\ldots\ldots\ldots\; (\text{A}).$$

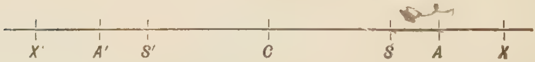



Fig. 26.

Adding these two results, we have

$$SA + SA' = e\,(AX + A'X).$$

But $$SA + SA' = AA' = 2a$$

and $$AX + A'X = A'X' + A'X = XX' = 2CX.$$

$$\therefore\quad 2a = 2e\,.\,CX\,;$$

$$\therefore\quad CX = a/e \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (4).$$

(ii.) Again, from (A), by subtracting, we have

$$SA' - SA = e\,(A'X - AX).$$

$$\therefore\quad SA' - S'A' = e\,(A'X - AX),$$

*i.e.*, $$SS' = e\,.\,AA' \quad\text{or}\quad 2CS = 2ae,$$

$$\therefore\quad CS = ae \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (5).$$

Cor. $$CX\,.\,CS = a/e \times ae = a^2 = CA^2.$$

62. **Latus rectum.**—Definition.—A chord $LSL'$ drawn through a focus perpendicular to the axis is called the **latus rectum.** Its length is usually represented by $2l$.

*The semi-latus rectum* $l = b^2/a$.

For the chord $LSL'$ is, of course, bisected at $S$, and, since $CS = ae$, the coordinates of $L$ are $ae$, $l$.

Hence $\dfrac{a^2e^2}{a^2} + \dfrac{l^2}{b^2} = 1$.

$$\therefore \quad \frac{l^2}{b^2} = 1 - e^2 = \frac{b^2}{a^2};$$

$$\therefore \quad \boldsymbol{l = b^2/a} \quad \ldots\ldots\ldots \ (\mathbf{6}).$$

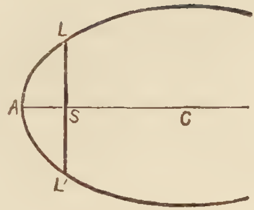


Fig. 27.

*Example.—An ellipse has a focus at* (1, 1), *directrix* $3x + 4y - 32 = 0$, *and eccentricity* $\frac{1}{3}$; *find the coordinates of* $A$, $A'$, *and* $C$, *the lengths of the axes, and the position of the other focus and directrix. Find also the equation of the ellipse.*

In the first place the equation is clearly

$$\left\{(x-1)^2 + (y-1)^2\right\}^{\frac{1}{2}} = \tfrac{1}{3}\,\frac{3x+4y-32}{\sqrt{25}}.$$

Thus $\qquad 225\left\{(x-1)^2+(y-1)^2\right\} = (3x+4y-32)^2$;

*i.e.*, $\qquad 216x^2 - 24xy + 209y^2 - 258x - 194y - 574 = 0$.

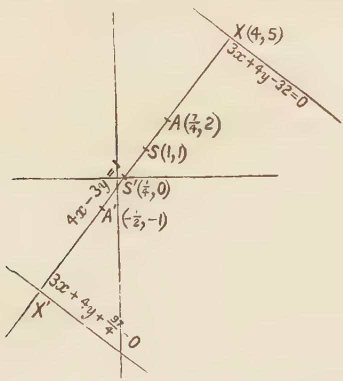


Fig. 28.

Again, $A$ and $A'$ divide $SX$ internally and externally in a given ratio 1:3. So we must find $X$.

Now, $SX$ being perpendicular to $3x+4y-32=0$, its equation is of the form $\qquad 4x - 3y + k = 0$,

and, since it passes through (1, 1) it is $\;4x - 3y = 1$.

$X$ is where this line meets $3x+4y-32=0$.

Hence its coordinates are 4, 5.

Hence $A$ is $x=\dfrac{3\cdot1+1\cdot4}{4}=\frac{7}{4}$, $y=\dfrac{3\cdot1+1\cdot5}{4}=2$. (Part I., § 3)

Similarly $A'$ is $(-\frac{1}{2}, -1)$.

Again, $C$ is the mid point of $AA'$, and is thus $(\frac{5}{8}, \frac{1}{2})$.

To find the axes we have

$$CA=\sqrt{(\tfrac{7}{4}-\tfrac{5}{8})^2+(2-\tfrac{1}{2})^2};$$

$$\therefore\quad a=\tfrac{15}{8}\quad\text{and}\quad b=a\sqrt{1-e^2}=\tfrac{15}{8}\sqrt{1-(\tfrac{1}{3})^2}=\tfrac{5}{4}\sqrt{2}.$$

If the other focus be $(x_1, y_1)$, then $C$ is the mid-point of (1, 1) and $(x_1, y_1)$. Thus
$$\left.\begin{aligned} x_1+1&=2\cdot\tfrac{5}{8}=\tfrac{5}{4};\quad\therefore\quad x_1=\tfrac{1}{4}\\ y_1+1&=2\cdot\tfrac{1}{2}=1;\quad\therefore\quad y_1=0\end{aligned}\right\}.$$

Therefore the other focus is $(\frac{1}{4}, 0)$.

The other directrix is parallel to the former, and its distance from the centre is equal in magnitude, but opposite in sign, to the distance of the former. Thus its equation is $3x+4y=k'$, and we have

$$\frac{3x+4y-k'}{5}=-\frac{3x+4y-32}{5}\quad\text{where } x=\tfrac{5}{8}\text{ and } y=\tfrac{1}{2}.$$

Hence $k'=-\frac{97}{4}$, and the other directrix is $3x+4y+\frac{97}{4}=0$.

## Exercises.

21. With the figure of § 56 show that

$$SB=AC,\quad CS=\sqrt{a^2-b^2}.$$

22. The semi-axes of an ellipse are 4 and 3 respectively. Find the lengths of the radii vectores inclined at angles 30°, 45°, and 60° to the major axis.

23. Find the eccentricity and the latus rectum of the same ellipse.

24. Find the equation of the ellipse whose focus is at (1, 2), directrix $x+y+1=0$, and eccentricity $\frac{1}{2}$. Find also the length of its latus rectum.

[$l=e\times$ perpendicular from focus on directrix.]

25. Find the lengths of the major and minor axes of the ellipse in Ex. 24.

26. Find the coordinates of the extremities of the major axis of the same ellipse.

27. Find the other focus and directrix of the above ellipse.

28. Find the equation of the ellipse whose focus is $(\frac{1}{4}, 0)$, directrix $3x+4y+\frac{47}{4}=0$, and eccentricity $\frac{1}{3}$.

29. Being given the positions and lengths of the semi-axes of an ellipse, show how to find the foci and directrices.

**63. The sum of the focal distances of any point on the curve is equal to the major axis.**

Let the coordinates of $P$ be $x, y$, $PN$ its ordinate, and $PK$, $PK'$ the perpendiculars to the directrices. Then

$$SP = ePK = eNX$$
$$= e(CX + x)$$
$$= e\left(\frac{a}{e} + x\right), \quad \text{since} \quad CX = \frac{a}{e};$$

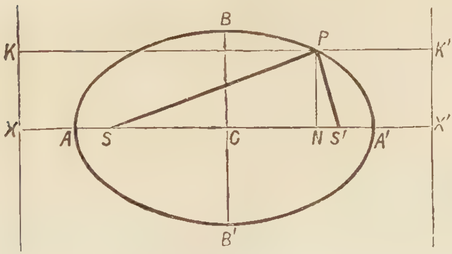



Fig. 29.

$$\therefore \quad SP = a + ex.$$

Similarly, $\quad S'P = ePK' = eNX' = e(CX' - CN)$

$$= e\left(\frac{a}{e} - x\right) = a - ex,$$

and hence $\quad SP + S'P = a + ex + a - ex;$

$$\therefore \quad SP + S'P = 2a \quad \text{.......................} \quad (7).$$

64. **Conversely, if a point moves in a given plane so that the sum of its distances from two points in that plane is constant, it describes an ellipse.**

Take the line joining the given points $S, S'$ as the axis of $x$ and its middle point $C$ as the origin. Then the coordinates of $S$, $S'$ may be represented by $(c, 0)$ and $(-c, 0)$.

If $(x, y)$ be a point $P$ on the curve, we have

$$SP + S'P = \text{constant} = 2a, \text{ say},$$
$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a,$$

and $\quad SP + S'P > SS' \quad \text{or} \quad a > c.$

Transposing, we get

$$\sqrt{(x-c)^2+y^2}-2a = -\sqrt{(x+c)^2+y^2},$$

and, on squaring, we obtain

$$(x-c)^2+y^2-4a\sqrt{(x-c)^2+y^2}+4a^2 = (x+c)^2+y^2.$$

Hence, on rearranging and dividing across by 4, we obtain

$$a\sqrt{(x-c)^2+y^2} = a^2-cx,$$

and, on squaring again,

$$a^2(x-c)^2+y^2a^2 = a^4-2a^2cx+c^2x^2,$$

*i.e.*

$$x^2(a^2-c^2)+y^2a^2 = a^4-a^2c^2 = a^2(a^2-c^2)$$

or

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2} = 1$$

Since $a > c$, this is an ellipse with the points $(c, 0)$, $(-c, 0)$ as foci. (See § 61.)

65. **Mechanical description of the ellipse.**—From § 64 we can at once deduce the easiest method for the mechanical description of an ellipse.

Take a thread $SPS'$ and fasten its two ends $S$, $S'$ firmly to two nails fixed on the paper. Then place a pencil vertically against the thread so as to keep it always tight. Move the pencil about, and it will describe an ellipse whose foci are $S$ and $S'$ and whose major axis

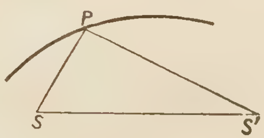



Fig. 30.

$$= SP+S'P = \text{length of thread.}$$

NOTE.—To describe the lower portion of the ellipse the whole thread must be brought to the lower side of $SS'$ and the pencil placed against the upper side of the thread.

**Exercises.**

30. Find the simplest form of equation of the locus of a point that moves so that the sum of its distances from two points $S$, $S'$ is 10 where $SS' = 8$.

31. Find the eccentricity of the curve in Ex. 30, and the length of its semi-latus rectum.

66. **Ellipses are not all of the same shape.** (Compare Corollary in § 39.)

Suppose, after describing the ellipse $APA'$ with thread $SPS'$ and foci $S$ and $S'$, we fasten the ends of the threads to points $V$, $V'$ on $SS'$, where $SV = S'V'$.

Then it is clear that in general $VP + V'P$ is not equal to $SP + S'P$, and consequently the curve described by the thread with $V$ and $V'$ as foci *will not pass through* $P$.

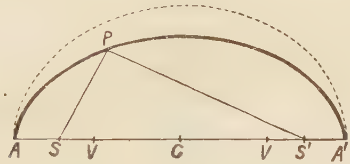



Fig 31.

Similarly it will not pass through *any** other point of the ellipse $APA'$ except $A$ and $A'$.

Hence the curve described will be another ellipse with $AA'$ as major axis, lying entirely within or entirely without† ellipse $APA'$ except at $A$ and $A'$, where they touch.

Hence these ellipses are not of the same shape, which proves the proposition.

67. **The circle is a limiting case of an ellipse.**

If the two fixed points of § 64 coincide, the locus is clearly a circle of radius $a$.

Hence we infer that,

*if the foci of an ellipse coincide, it becomes a circle,*

having its centre at the point of coincidence.

This also follows from the equation of § 64, for, if the two points coincide, $c = 0$ and the equation becomes

$$x^2 + y^2 = a^2,$$

which represents a circle of radius $a$ having its centre at the origin.

Also, since $CS = ae$, and in the case of the circle $CS = 0$, we see that

*the eccentricity of a circle is zero.*

Also $CX = \dfrac{a}{e} = \infty$ for a circle, and therefore its

*directrices are at an infinite distance from the centre.*

* The curves therefore cannot intersect.

† Within if $V$ and $V'$ lie in $SS'$ produced, without if $V$ and $V'$ lie in $SS'$. This can be proved by considering the position of $P$ when it is on the minor axis, *i.e.*, when the two parts of the string make equal angles with $SS'$.

**68. To show that the straight line $y=mx+c$ meets the ellipse $x^2/a^2+b^2/y^2=1$ in two points, real or imaginary; and to find the condition that the line should touch the ellipse.**

To find the points common to the two loci, we must solve, simultaneously, the equations

$$y=mx+c \quad \text{and} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Putting $$y=mx+c$$

in the second equation, we have the quadratic for $x$

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1,$$

or $$x^2\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right)+\frac{2mc}{b^2}.x+\frac{c^2}{b^2}-1=0.$$

This equation gives two values of $x$, and for each value of $x$ we get one value of $y$ from

$$y=mx+c\,;$$

so there are two points of intersection.

The values of $x$ are real, coincident, or imaginary, according as

$$\frac{m^2c^2}{b^4}-\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right)\left(\frac{c^2}{b^2}-1\right)\gtreqless 0,$$

(*Tut. Alg.*, II., § 159)

*i.e.*, according as $$\frac{m^2}{b^2}+\frac{1}{a^2}-\frac{c^2}{a^2b^2}\gtreqless 0,$$

or as $$a^2m^2+b^2-c^2\gtreqless 0.$$

Therefore, if $$c=\pm\sqrt{a^2m^2+b^2},$$

the two points coincide, and hence the lines

$$\boldsymbol{y=mx\pm\sqrt{a^2m^2+b^2}} \quad \ldots\ldots\ldots\ldots\ldots \quad \textbf{(8)}$$

touch the ellipse for all values of $m$. The double sign shows that there are two tangents parallel to $y = mx$.

Cor. Since the ellipse is a closed curve, it is evident that no real line can meet it at an infinite distance.

This also follows at once from the quadratic for $x$, for, if it had an infinite root, we should have

$$\frac{1}{a^2} + \frac{m^2}{b^2} = 0, \quad (\textit{Tut. Alg.}, \text{II.}, \S\ 166)$$

leading to $$m = \pm \frac{b}{a}\sqrt{-1},$$

which is imaginary.

**Exercises.**

32. Find, from first principles, the condition that $y = 3x + c$ should touch the ellipse $x^2 + 4y^2 = 1$, and find the coordinates of the point of contact.

33. Find the equations of the tangents to $x^2 + 4y^2 = 1$ that make an angle of 45° with the major axis.

34. Find the coordinates of the points of contact of the tangents to $x^2/a^2 + y^2/b^2 = 1$ making an angle $\alpha$ with the major axis, and show that the line joining these points passes through the centre.

**69. The equation of the ellipse is of the second degree whatever be the axes of reference, and, if the terms of the second degree be, as usual,**

$$ax^2 + 2hxy + by^2,$$

**then** $$ab > h^2.$$

In its simplest form the equation is of the form

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$$

Now, to change to any other axes, we have to replace $x$ and $y$ by linear functions of the new coordinates (Part I., § 35); hence the new equation is of the form

$$\frac{(l_1x + m_1y + n_1)^2}{\alpha^2} + \frac{(l_2x + m_2y + n_2)^2}{\beta^2} = 1.$$

The terms of the second degree are

$$\frac{(l_1x + m_1y)^2}{\alpha^2} + \frac{(l_2x + m_2y)^2}{\beta^2}$$

viz., the *sum* of two squares, so that the factors of the expression $ax^2+2hxy+by^2$ must be imaginary, and hence $ab>h^2$.

We can easily verify this by putting in for $a$, $h$, $b$ their values as determined by comparing the two equations, thus,

$$a=\frac{l_1^2}{\alpha^2}+\frac{l_2^2}{\beta^2},\quad b=\frac{m_1^2}{\alpha^2}+\frac{m_2^2}{\beta^2},\quad h=\frac{l_1m_1}{\alpha^2}+\frac{l_2m_2}{\beta^2}.$$

Hence

$$(ab-h^2)=\left(\frac{l_1^2}{\alpha^2}+\frac{l_2^2}{\beta^2}\right)\left(\frac{m_1^2}{\alpha^2}+\frac{m_2^2}{\beta^2}\right)-\left(\frac{l_1m_1}{\alpha^2}+\frac{l_2m_2}{\beta^2}\right)^2$$

$$=\left(\frac{l_1m_2-l_2m_1}{\alpha\beta}\right)^2,$$

which is clearly positive, since it is a perfect square.

70. The above result can also be shown as follows when the axes are rectangular.

Let $(x', y')$ be a focus, and $x\cos\alpha+y\sin\alpha-p=0$ the corresponding directrix; then, if $(x, y)$ be on the curve, we have

$$\sqrt{(x-x')^2+(y-y')^2}=e\,(x\cos\alpha+y\sin\alpha-p),$$

or, on squaring,

$$(x-x')^2+(y-y')^2=e^2\,(x\cos\alpha+y\sin\alpha-p)^2.$$

The terms of the second degree are

$$x^2\,(1-e^2\cos^2\alpha)-2xye^2\sin\alpha\cos\alpha+y^2\,(1-e^2\sin^2\alpha);$$

so that, with the usual notation,

$$a=1-e^2\cos^2\alpha,\quad b=1-e^2\sin^2\alpha,\quad h=-e^2\sin\alpha\cos\alpha,$$

whence

$$ab-h^2=1-e^2,$$

which is positive, since $e<1$.

We shall afterwards see (Chap. VI.) that, if $ab>h^2$, the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

*always* represents an ellipse. We have here proved the converse of this proposition.

**Exercise.**

35. Take the curve $x^2/9+y^2=1$ (axes supposed rectangular). Find its equation when the origin is transferred to the point 1, 1, and then when, further, the axes are turned through 30°. Confirm § 69 by showing that, in each of the three stages, $ab>h^2$.

## MISCELLANEOUS EXERCISES ON CHAP. IV.

36. Being given a focus $S$ of an ellipse and the corresponding vertex, show that the locus of the extremities of the minor axis is a parabola having its focus at $S$.

[Use the relation $SB = AC$.]

37. Prove that the line $lx + my = 1$ will touch the ellipse $x^2/a^2 + y^2/b^2 = 1$ provided $a^2l^2 + b^2m^2 = 1$.

38. Find the coordinates of the middle point of the portion of the straight line $x + y = 2$ intercepted by the ellipse $3x^2 + 2y^2 = 6$.

39. Prove that, if $lx + my = 1$ meets $x^2/a^2 + y^2/b^2 = 1$ in real points, the coordinates of the middle point of the intercepted portion are

$$\frac{a^2l}{a^2l^2 + b^2m^2}, \quad \frac{b^2m}{a^2l^2 + b^2m^2}.$$

40. Of two circles one completely encloses the other. Show that the locus of the centre of a circle touching the inner one externally and the outer one internally is an ellipse, having its foci at the centres of the two circles.

[See that the sum of the distances is constant.]

41. Find the eccentricity, the length of the latus rectum, and the coordinates of the foci of the ellipse $4(x-1)^2 + 3y^2 = 4$, and represent the curve in a figure.

42. Find the equation of the lines joining the origin to the points in which the line $x \cos\alpha + y \sin\alpha - p = 0$ meets the ellipse $x^2/a^2 + y^2/b^2 = 1$; and hence show that, if the intercepted portion of the chord subtends a right angle at the centre, it always touches a circle of radius $\dfrac{ab}{\sqrt{a^2 + b^2}}$ concentric with the ellipse.

43. Which of the lines $3x + 4y + 5 = 0$, $8x + 9y - 12 = 0$ lies nearer the origin? Does either of them meet the curve $2x^2 + 4y^2 = 3$?

44. An endless string passes round two fixed cylindrical pegs and round a cylindrical pencil. Taking into account the dimensions of the cylinders, but supposing them to have equal radii, show that, if the string be kept tight and the pencil moved, the pencil will describe an ellipse if the tracing point is at its centre.

45. If $V$ be the mid-point of a chord $PP'$ of an ellipse, $VK$ the perpendicular on a directrix, and $S$ the corresponding focus, then

$$SP + SP' = 2eVK.$$

Deduce that the locus of the middle point of a chord, when the sum of the distances of its extremities from a focus is constant, is a line parallel to the minor axis.

46. If $S$, $S'$ are the foci of an ellipse, and $P$ any point on the curve show that $\tan \frac{1}{2}PSS' \tan \frac{1}{2}PS'S = (1-e)/(1+e)$.

Conversely, if the base of a triangle be given and the product of the tangents of half its base angles, show that its vertex lies on an ellipse having the ends of the base for foci.

$$\left[\text{Use} \quad \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.\right]$$

47. Find the eccentricities, foci, and directrices of the ellipses

(i.) $x^2+4y^2 = 6x$; (ii.) $4x^2+4y^2 = 5x$.

48. Prove that, if a bar of given length moves with its extremities on two fixed straight lines at right angles, any point marked on it describes an ellipse.

49. Find the coordinates of the foci, and the equations of the directrices, of the conic $x^2+y^2 = e^2(x\cos\alpha + y\sin\alpha - p)^2$.

50. Find the equation referred to rectangular axes of an ellipse of eccentricity $\frac{1}{\sqrt{2}}$, which passes through the origin, and has its focus at some point of $x^2+y^2-ax = 0$, the corresponding directrix being

$$x+y-a = 0.$$

51. Find the equation of the ellipse $x^2/a^2+y^2/b^2 = 1$ when the point $(-a, 0)$ is taken for origin, the directions of the axes being unaltered.

Deduce that the equation $y^2 = 2px+qx^2$ represents an ellipse if $q$ be negative. Show, further, that the latus rectum is of length $2p$, and that the eccentricity is $\sqrt{1-q}$. What does the equation represent when $q = 0$?

52. If $x_1$, $x_2$ be the abscissæ of the points $P$, $Q$ on the ellipse $x^2/a^2+y^2/b^2 = 1$, and $P_1$, $Q_1$ be points on the major axis whose abscissæ are $ex_1$ and $ex_2$ respectively, prove that $PQ_1 = P_1Q$.

# CHAPTER V.

## THE HYPERBOLA.

71. **Hyperbola.** — DEFINITIONS. — A **hyperbola** is the locus of a point which moves so that its distance from a fixed point bears a constant ratio of greater inequality to its distance from a fixed straight line.

The fixed point is called a **focus**, the fixed line the corresponding **directrix**, and the constant ratio the **eccentricity**. For a hyperbola, therefore, the eccentricity $e$ is greater than 1.

### 72. To find the equation of a hyperbola.

[The method is exactly the same as for the ellipse.]

Let $S$ be a focus and $XK$ the corresponding directrix, $SX$ being perpendicular to $XK$.

Take $X$ for origin and let $S$ be $(p, 0)$; then, as in the case of the ellipse (§ 54), we have

$$SP = e PK;$$

$$\therefore \quad SP^2 = e^2 PK^2;$$

$$\therefore \quad (x-p)^2 + y^2 = e^2 x^2$$

or $x^2(1-e^2) + y^2 - 2px + p^2 = 0$, the only difference being that now $e > 1$.

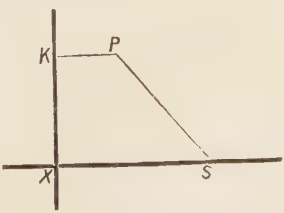



Fig. 32.

**73. Reduction of the equation of the hyperbola to the form** $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

[Here again the procedure is the same as in the case of the ellipse.]

The equation just found may be written

$$(1-e^2)\left\{x^2-2x\cdot\frac{p}{1-e^2}\right\}+y^2+p^2=0,$$

or, on completing the square with respect to $x$,

$$(1-e^2)\left\{x-\frac{p}{1-e^2}\right\}^2+y^2+p^2=\frac{p^2}{1-e^2};$$

but, as $e>1$, we write this

$$(e^2-1)\left(x+\frac{p}{e^2-1}\right)^2-y^2-p^2=\frac{p^2}{e^2-1}.$$

Now let us take as new origin the point $[-p/(e^2-1), 0]$;

then $$(e^2-1)x^2-y^2=\frac{p^2}{e^2-1}+p^2=\frac{p^2e^2}{e^2-1}\quad\text{(Pt. I., §31)}$$

or $$x^2-\frac{y^2}{e^2-1}=\frac{p^2e^2}{(e^2-1)^2}.$$

Now put $$p^2e^2/(e^2-1)^2=a^2;$$

then, on dividing across by $a^2$, the above becomes

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1,$$

and, on making $$a^2(e^2-1)=b^2,$$

this is $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad\text{.................. }\mathbf{(1)}.$$

Cor. $$e^2=1+\frac{b^2}{a^2}.$$

*Example.—Find the equation of a hyperbola whose directrix is* $2x+y=1$, *focus* (1, 2), *and eccentricity* $\sqrt{3}$.

Here, if $P(x, y)$ be a point on the curve, we have

$$SP^2=(x-1)^2+(y-2)^2,$$

$$PK = \text{perpendicular from } (x, y) \text{ on } 2x+y-1=0$$
$$=\frac{2x+y-1}{\sqrt{5}}.$$

Hence the equation is

$$(x-1)^2+(y-2)^2=\tfrac{3}{5}(2x+y-1)^2 \quad\text{since}\quad e^2=3,$$

or, on reduction, $7x^2+12xy-2y^2-2x+14y-22=0$.

**Exercises.**

1. Find the equation of a hyperbola whose focus is the point $(0, 2)$, directrix $x+y=1$, and eccentricity $\sqrt{2}$.

2. The focus being $(\sqrt{a^2+b^2}, 0)$, the directrix $x = \dfrac{a^2}{\sqrt{a^2+b^2}}$, and eccentricity $\dfrac{\sqrt{a^2+b^2}}{a}$, show that the equation of the hyperbola is

$$x^2/a^2 - y^2/b^2 = 1.$$

3. If the focus of a hyperbola is at the origin, the directrix the line $x+2=0$, and the eccentricity 2, find the equation.

4. Show that the equations $x^2-y^2/2=1$, $x^2/2-y^2/3=2$, $3x^2-4y^2=5$, $ax^2-by^2=c$ ($a$, $b$, $c$ being positive) all denote hyperbolas; and find the lengths of the semi-axes.

[Each equation must be reduced to the form $\dfrac{x^2}{\alpha^2}-\dfrac{y^2}{\beta^2}=1$, *e.g.*, if $2x^2-y^2=3$, we have $\dfrac{2}{3}x^2-\dfrac{y^2}{3}=1$ or $\dfrac{x^2}{3/2}-\dfrac{y^2}{3}=1$, and the curve denotes a hyperbola whose semi-axes are $\sqrt{\tfrac{3}{2}}$, $\sqrt{3}$.]

5. Find the eccentricities of the hyperbolas in Ex. 4.

[Use $e^2=1+b^2/a^2$.]

**74. Shape of the curve.**—As in the case of the ellipse (§ 56), the equation $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

gives us a good idea as to the shape of the curve.

From the equation we have

$$y^2 = b^2\left(\frac{x^2}{a^2}-1\right),$$

and since $y^2$ is positive, we see that $x^2/a^2 \nless 1$, so that $x$ cannot be *numerically* less than $a$.

Also
$$x^2 = a^2\left(1+\frac{y^2}{b^2}\right),$$

but this gives us no limitation for $y$, and in fact $y$ can have any value.

Again, from the equations

$$y = \pm\frac{b}{a}\sqrt{x^2-a^2} \quad\text{and}\quad x = \pm\frac{a}{b}\sqrt{y^2+b^2}.$$

we infer (1) $x$ cannot be numerically less than $a$;

(2) $x = \pm a$ makes $y = 0$;

(3) any value of $x$ numerically greater than $a$ gives two equal and opposite values of $y$;

(4) any value of $y$ gives two equal and opposite values of $x$.

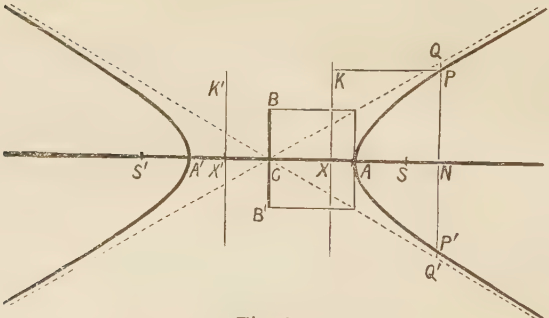



Fig. 33.

Hence the curve is symmetrical, with respect to both axes; and lies entirely outside the lines $x = +a$ and $x = -a$.

Further, every chord parallel to one axis is of course bisected by the other.

If $A$, $A'$ be taken on the axis of $x$, so that

$$CA = CA' = a,$$

and $B$, $B'$ on the axis of $y$, so that

$$CB = CB' = b,$$

then $AA'$, $BB'$ are called the axes. Sometimes the words "major" and "minor" are applied to them, as in the case of the ellipse; but there is an important difference in the two cases, for in the ellipse $B$ and $B'$ are on the curve, while in the hyperbola they are not. Hence $AA'$ is more usually called the *transverse* axis, and $BB'$ the *conjugate* axis. Besides, as $b^2 = a^2(e^2 - 1)$, when $e > \sqrt{2}$, $b > a$; so that the name *minor* is clearly unsuitable. As before, $C$ is called the centre.

To obtain the shape properly here, it is necessary to use the polar equation, which we shall now obtain.

75. **Polar equation of the hyperbola, the centre being the pole.**

If we write $x = r\cos\theta$ and $y = r\sin\theta$ in the equation

$$x^2/a^2 - y^2/b^2 = 1,$$

we at once obtain $$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$= \frac{1}{a^2} - \sin^2\theta\left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

two different forms of the polar equation required.

76. **Shape of the curve deduced from the polar equation.**

When $\theta = 0$, $r = \pm a$; as $\theta$ increases, $1/r$ diminishes numerically, and so the curve goes further away from $C$ continually; $r$ becomes infinite when $1/r = 0$, *i.e.*, when

$$\cos^2\theta/a^2 = \sin^2\theta/b^2 \quad \text{or} \quad \tan\theta = b/a.$$

Thus the radius vector that makes an angle $\tan^{-1} b/a$ with the axis meets the curve at an infinite distance.

Hence the part of the curve in the positive quadrant is as shown in the figure. The radius vector $r$ becomes greater and greater as its direction approaches the line for which $\theta = \tan^{-1} b/a$, and, as the curve is symmetrical in all the other quadrants, it can be drawn completely.

It must be carefully noted that, as $1/r^2$ is negative for values of $\theta$ for which $\tan\theta$ is numerically greater than $b/a$, *i.e.*, for values of $\theta$ between $\tan^{-1} b/a$ and the supplement of this angle, there is no portion of the curve between the two corresponding lines, for the value of $r$ corresponding to values of $\theta$ between these limits is imaginary. (See Fig. 33.)

The values of $\theta$ which make $r$ infinite are given by

$$\tan\theta = \pm b/a,$$

and the lines through the centre which thus meet the curve at an infinite distance are known as asymptotes. This, it must be observed, is not the definition of asymptotes which will be given later.

77. **Comparison between ellipse and hyperbola.**

The reader, while noting the points of similarity between the ellipse and the hyperbola, must carefully notice the points of difference.

(i.) The ellipse is a closed curve, while the hyperbola extends to an infinite distance.

(ii.) The ellipse meets both its axes in real points; the hyperbola meets one axis only in real points.

(iii.) The centre and a focus are on the same side of the corresponding directrix in the ellipse, but on opposite sides in the hyperbola.

*Example* (i.). *In the hyperbola whose transverse and conjugate axes are* 3 *and* 2 *respectively, find the radii vectores making angles* 30° *and* 60° *with the major axis.*

The polar equation is $\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}$. When $\theta = 30°$, we have

$$\frac{1}{r^2} = \frac{\frac{3}{4}}{9} - \frac{\frac{1}{4}}{4} = \frac{1}{12} - \frac{1}{16} = \frac{1}{48} \quad \text{or} \quad r = 4\sqrt{3}.$$

When $\theta = 60°$, we have $\frac{1}{r^2} = \frac{\frac{1}{4}}{9} - \frac{\frac{3}{4}}{4} = -\frac{23}{144}$ or $r = \sqrt{-\frac{144}{23}}$.

Thus $r$ is imaginary in this case, as it should be, for the asymptote makes an angle $\tan^{-1}\frac{2}{3}$ with the major axis, and, as this is less than 60°, the latter line does not meet the curve in real points.

*Example* (ii.). **The sum of the squares of the reciprocals of two radii vectores at right angles is constant.**

Here, just as in the ellipse, if the extremities be $(r, \theta)$ and $(r', \overline{\theta + \frac{1}{2}\pi})$,

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2},$$

$$\frac{1}{r'^2} = \frac{\cos^2\left(\theta + \frac{\pi}{2}\right)}{a^2} - \frac{\sin^2\left(\theta + \frac{\pi}{2}\right)}{b^2} = \frac{\sin^2\theta}{a^2} - \frac{\cos^2\theta}{b^2};$$

$$\therefore \quad \frac{1}{r^2} + \frac{1}{r'^2} = \frac{1}{a^2} - \frac{1}{b^2};$$

but it must be carefully noted that $r$ and $r'$ may be either or both imaginary.

### Exercises.

6. Sketch roughly, in one diagram, the curves (i.) $x^2/9 - y^2 = 1$, (ii.) $4x^2 - y^2 = 1$, (iii.) $4x^2 - 9y^2 = 1$.

7. Find the lengths of the radii vectores of the curves in Ex. 6 that make angles of 30° and 45° with the transverse axes.

8. Sketch, in one diagram, the curves $x^2 - y^2 = 9$, $x^2 - 4y^2 = 1$; find the tangent of the angle the common radius vector makes with the axis of $x$, and find the length of that radius vector.

**78. Equation of hyperbola referred to axes parallel to the principal axes.**

We shall now give a few examples illustrating the tracing of hyperbolas when the axes of coordinates are parallel to, but not both coincident with, the axes of the curve (*cf.* § 57).

*Example* (i.). *Trace the hyperbola*

$$4(x-1)^2-9(y+3)^2=9.$$

The equation can be written

$$\frac{(x-1)^2}{\frac{9}{4}}-\frac{(y+3)^2}{1}=1.$$

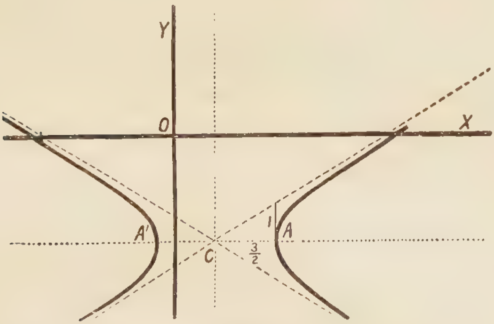



Fig. 34.

Transferring the origin to $(1, -3)$, the equation becomes

$$\frac{x^2}{(\frac{3}{2})^2}-\frac{y^2}{1^2}=1,$$

which is a hyperbola with semi-axes whose lengths are $\frac{3}{2}$ and 1. The curve is as drawn.

The curve meets the original axis of $x$ where $y=0$, and $4(x-1)^2=9+81$, or $x=1\pm4\frac{3}{4}$, approx. It meets the original axis of $y$, where $x=0$, and $9(y+3)^2=4-9=-5$, *i.e.*, in imaginary points.

The student should find additional points on the curve by giving $x$ or $y$ other values, as in § 43.

*Example* (ii.). *Trace the curve*

$$9x^2-2y^2-18x-4y+25=0.$$

Collecting terms [as in § 57, Example (ii.)] so as to make the terms

in $x^2$ and $x$ a perfect square, and similarly those in $y^2$ and $y$, we have

$$9(x^2-2x+1)-2(y^2+2y+1)+18=0,$$

or

$$9(x-1)^2-2(y+1)^2=-18,$$

or

$$\frac{(y+1)^2}{9}-\frac{(x-1)^2}{2}=1.$$

Transfer the origin to the point $(1, -1)$, and we get

$$\frac{y^2}{9}-\frac{x^2}{2}=1.$$

The curve is therefore a hyperbola whose transverse axis is along the *new* axis of $y$, the semi-axes being 3, $\sqrt{2}$ in length.

The intercepts on the original axes are given by $9x^2-18x+25=0$ and $-2y^2-4y+25=0$. The former are imaginary; the latter are $-1\pm\sqrt{\frac{27}{2}}$ or $-1\pm 3{\cdot}7$, approximately.

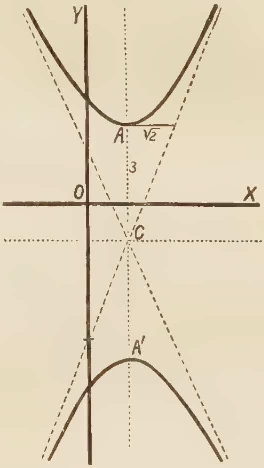



Fig. 35.

### Exercises.

Trace the curves:—

9. $\frac{(x-1)^2}{9}-\frac{(y+\frac{1}{2})^2}{4}=1.$

10. $(x+1)^2-4(y-3)^2=8.$

11. $4x^2-y^2+16x+2y-13=0.$

12. $9x^2-y^2-18\sqrt{3}x+36=0.$

13–16. Find the eccentricities of the curves in Exx. 9–12. Find also the coordinates of the ends of the transverse axes with reference to the original axes of coordinates.

79. **The hyperbola has a second focus and a second directrix.**

Since the curve is symmetrical with respect to the axes, if we take $CS'=CS$ (Fig. 33) and $CX'=CX$, and draw $X'K'$ at right angles to $ACA'$, then, just as in the ellipse, $S'$ is a second focus, and $X'K'$ the corresponding directrix. The eccentricity is, of course, the same for both foci.

**Caution.**—It must be noticed carefully that the two different branches of the hyperbola constitute one and the same curve, and that with either focus and directrix we can obtain *both* branches of the curve, and not merely the branch surrounding that focus.

80. **To show that** $CS = ae$, $CX = a/e$.

Since $A$ and $A'$ are on the curve, we have

$$SA = eAX, \quad SA' = eA'X.$$

Adding these equations, we obtain

$$SA + SA' = e(AX + A'X) \quad \text{or} \quad S'A' + SA' = e(AX + A'X);$$

$$\therefore \quad SS' = eAA'.$$

Therefore, since $SS' = 2CS$ and $AA' = 2CA$,

we have $$CS = eCA \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$


Fig. 36.

Again, on subtraction, we have

$$SA' - SA = e(A'X - AX) = e(A'X - A'X') \quad \text{or} \quad AA' = eXX',$$

*i.e.* $$CA = e.CX \quad\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Cor. $$CS.CX = a^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

81. **Latus rectum.** — Definition. — A chord $LSL'$ through the focus perpendicular to the axis is called the **latus rectum.**

**The semi-latus rectum** $= b^2/a$ ………………… **(7).**

The proof is exactly similar to that for the ellipse. (§ 62)

*Example.—Find the equation of the hyperbola whose focus is* (1, 1), *directrix* $3x + 4y - 32 = 0$, *eccentricity* 3. *Also find the coordinates of the ends of the transverse axis, of the centre, and of the other focus. Find the two semi-axes.*

The equation is easily seen to be, on squaring,

$$25\left\{(x-1)^2 + (y-1)^2\right\} = 9(3x + 4y - 32)^2,$$

reducing to $56x^2 + 216xy + 119y^2 - 1678x - 2254y + 9166 = 0$.

As in Example, p. 59, the equation of $SX$ (in the standard figure) is $$4x - 3y - 1 = 0,$$

and hence $X$ is the point (4, 5). Then $A$ and $A'$ divide $SX$ internally and externally in the ratio 3 : 1, and hence, by the usual formula,

$$A \text{ is } (\tfrac{13}{4}, 4) \quad \text{and} \quad A' \text{ is } (\tfrac{11}{2}, 7).$$

The centre is the mid-point of $AA'$, and hence it is $(\frac{35}{8}, \frac{11}{2})$.

Again, since the centre is the mid-point of $SS'$, we easily find that

$$S' \text{ is } (\tfrac{31}{4}, 10).$$

Also $$4a^2 = AA'^2 = (\tfrac{11}{2} - \tfrac{13}{4})^2 + (7-4)^2 = (\tfrac{15}{4})^2.$$

Hence $a = \frac{15}{8}$ and $b = a\sqrt{e^2 - 1} = \frac{15}{8}.2\sqrt{2} = \frac{15}{4}\sqrt{2}$.

**Exercises.**

17. The semi-axes of a hyperbola are 4 and 7 units in length, the latter being the conjugate semi-axis: find the latus rectum, the eccentricity, the distance of the foci from the centre, and the distance of the directrices from the centre.

18. If $SX = p$, show that $l = pe$.
If the focus is $(1, 1)$, the directrix $5x+12y+9=0$, and the eccentricity 2, find the latus rectum.

19. Find the semi-axes of the hyperbola in Ex. 18.

20. In the hyperbola $\frac{x^2}{3}-\frac{y^2}{2}=1$, find the lengths of the radii vectores making angles 30° and 45° with the transverse axis.

21. Find the equation of the hyperbola having a focus at $(-1, 1)$, $x+y-2=0$ for corresponding directrix, and eccentricity $\sqrt{5}$.
Find the latus rectum of the hyperbola.

22. Prove that $l = a(e^2-1)$.

23. Draw the curve whose equation is $2x^2-3y^2=5$.

**82. The difference of the focal distances of any point on the curve is equal to the transverse axis.**

For we have, if $P$ be $(x, y)$ and $PN$ be drawn perpendicular to the axis,

$$SP = ePK = e(x-CX)$$

$$= e\left(x-\frac{a}{e}\right) = ex-a.$$

Similarly,

$$S'P = e\left(x+\frac{a}{e}\right) = ex+a.$$

$$\therefore\quad S'P-SP = 2a \ldots (8).$$

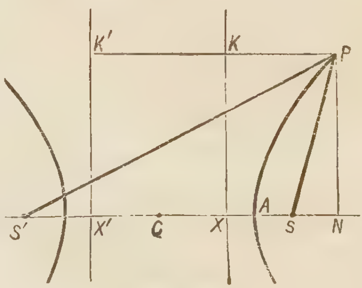



Fig. 37.

Thus, for the right-hand branch,

$$S'P-SP = 2a,$$

while, for the left-hand branch,

$$SP-S'P = 2a.$$

83. **Mechanical description of the hyperbola.**—From § 82 we can deduce a process for the mechanical description of a hyperbola.

Fasten one end of a flat ruler to the point $S'$, so that the ruler can revolve round $S'$ in the plane of the paper. Take a thread whose length is less than that of the ruler, and fasten one end of it to $S$, and the other end to the free end $T$ of the ruler. Then, if a pencil $P$ be pressed against the ruler and thread so as to keep the thread tight, the pencil, if placed as in diagram, will describe a part of the upper right-hand portion of the hyperbola, viz., from the vertex $A$ up to a point $P'$, where $SP'$ is the length of the string.

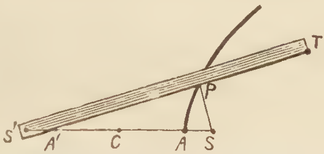


Fig. 38.

$$\text{For}\quad S'P - SP = (S'P + PT) - (SP + PT)$$
$$= S'T - (SP + PT)$$
$$= \text{length of ruler} - \text{length of string}$$
$$= \text{a constant quantity.}$$

The corresponding lower part of the right-hand portion of the hyperbola can be described by bringing the ruler below $S'S$; while the same parts of the left-hand portion of the hyperbola are drawn by fixing one end of the ruler to $S$ instead of $S'$, and one end of the thread to $S'$ instead of $S$.

### Exercises.

24. **Prove, as in § 64, that, if $SP - S'P$ be constant, $S$ and $S'$ being fixed points, the locus of $P$ is a hyperbola.**

25. Find the simplest form of equation of the locus of a point that moves so that the difference of its distances from two points $S$, $S'$ is 3 where $SS' = 8$.

26. Find the eccentricity of the curve in Ex. 25 and the length of its semi-latus rectum.

**84. To show that the line $y = mx + c$ meets the hyperbola $x^2/a^2 - y^2/b^2 = 1$ in two points, real or imaginary, and to find the condition of tangency.**

Here, proceeding exactly as in the ellipse, we find as the quadratic for the abscissæ,

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1,$$

or
$$x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) - 2\frac{mc}{b^2}x - \frac{c^2}{b^2} - 1 = 0.$$

Also each value of $x$ gives one value of $y$, by means of $y = mx + c$, and hence there are two points of intersection.

The values of $x$ are real, coincident, or imaginary, according as

$$\frac{m^2c^2}{b^4} + \left(\frac{1}{a^2} - \frac{m^2}{b^2}\right)\left(\frac{c^2}{b^2} + 1\right) \gtreqless 0,$$

*i.e.* $\quad \dfrac{1}{a^2} + \dfrac{c^2}{a^2b^2} - \dfrac{m^2}{b^2} \gtreqless 0$ or as $b^2 + c^2 - a^2m^2 \gtreqless 0.$

If
$$c = \pm\sqrt{a^2m^2 - b^2},$$

the two points coincide, and hence the lines

$$y = mx \pm \sqrt{a^2m^2 - b^2} \quad \text{..............} \quad \textbf{(9)}$$

touch the hyperbola for all values of $m$.

Cor. If $m < \dfrac{b}{a}$, the value of $c$ is imaginary, so that a tangent to a hyperbola cannot make a smaller angle with the axis than the asymptotes do.

If $m = \pm\dfrac{b}{a}$, $c = 0$, and the tangents are

$$y = \pm\frac{b}{a}x,$$

and these are, in fact, the asymptotes as found in § 76.

As they meet the curve only at an infinite distance, we see that **an asymptote may be looked on as a tangent whose point of contact is at infinity.**

85. **Asymptotes.**—DEFINITION.—A straight line which meets any curve in two coincident points at an infinite distance is called an **asymptote** of the curve.

It must be noted that we have not hitherto given a definition, but that the asymptotes as found already certainly possess this property.

**86. To find the asymptotes of the hyperbola**

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

If $y=mx+c$ be an asymptote, then the quadratic giving the abscissæ of the points of intersection must have both roots infinite.

Now the quadratic is

$$x^2\left(\frac{1}{a^2}-\frac{m^2}{b^2}\right)-2x\frac{mc}{b^2}-\frac{c^2}{b^2}-1=0;$$

and therefore the conditions for this are

$$\frac{1}{a^2}-\frac{m^2}{b^2}=0 \text{ and } \frac{2mc}{b^2}=0. \; (\textit{Tut. Alg.}, \text{II.}, \S\,167)$$

We at once deduce

$$m=\pm\frac{b}{a}, \quad c=0;$$

and thus the only asymptotes are the two already mentioned, namely, the two lines:

$$y=\pm\frac{b}{a}x \;\ldots\ldots\ldots\ldots\ldots\ldots\; \textbf{(10)},$$

and their joint equation is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=0.$

87. **Any line parallel to an asymptote will meet the curve in *one* point at infinity.**

One root of the equation for $x$ in last article is infinite provided

$$\frac{1}{a^2}-\frac{m^2}{b^2}=0,$$

*i.e.*, if

$$m=\pm\frac{b}{a},$$

which is the case if the line be parallel to one of the asymptotes.

*Example.*—**If $2\alpha$ be the angle between the asymptotes, then $e = \sec\alpha$.**

For
$$\tan\alpha = \frac{b}{a},$$
since the asymptotes are equally inclined to the axes; and therefore
$$\sec\alpha = \sqrt{1+\tan^2\alpha} = \sqrt{\frac{a^2+b^2}{a^2}} = e.$$

NOTE that $2\alpha$ is that angle between the asymptotes which encloses the curve; the other angle between them is $\pi - 2\alpha$.

88. **Rectangular or equilateral hyperbola.**

A hyperbola is said to be equilateral or rectangular when $b = a$, so that the equation reduces to $x^2 - y^2 = a^2$.

The reason for the latter name is that the asymptotes are at right angles, for they are now the pair of lines
$$x^2 - y^2 = 0 \quad \text{or} \quad x+y = 0, \quad x-y = 0.$$

COR. The rectangular hyperbola bears the same relation to the hyperbola that the circle does to the ellipse, for these particular forms are obtained by making the axes equal in the ellipse and hyperbola.

## Exercises.

27. Find from first principles the condition that $y = px + 3$ should touch the hyperbola $x^2 - 4y^2 = 9$, and find the coordinates of the point of contact.

28. Find the equations of the tangents to $x^2 - 40y^2 = 9$ that make an angle of 45° with the transverse axes.

29. Find whether the straight line $x + y = 2$ meets the hyperbola $x^2 - \frac{1}{2}y^2 = 1$ in real points or not.

30. Find the semi-axes of the hyperbola $2x^2 - 3y^2 = 5$, and show that it is touched by the straight line $y = x + \sqrt{\frac{5}{6}}$.

31. Show that the straight line $x + y = 0$ meets the curve
$$2x^2 + 3xy + y^2 + 3x + 2y = 0$$
in one point at infinity, and that the lines $x + y + 1 = 0$, $2x + y + 1 = 0$ both meet it in two points at infinity.

32. Find the value of $c$ in order that the line $y = x + c$ may touch the hyperbola whose focus is at (2, 0), directrix $2x - y + 3 = 0$, and eccentricity $\sqrt{2}$.

33. Show that the lines $x + 1 = 0$, $y + 3 = 0$ are asymptotes of the curve $xy + 3x + y = 0$.

34. The angle between asymptotes of a hyperbola is 60°: find its eccentricity.

[Use $e = \sec\alpha$.]

89. **The product of the perpendiculars on the asymptotes from any point of the curve is constant.**

The asymptotes are

$$x/a-y/b=0 \text{ and } x/a+y/b=0, \qquad (\S\ 86)$$

and the product of the perpendiculars from $(x_1, y_1)$ on them is

$$\frac{x_1/a-y_1/b}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} \cdot \frac{x_1/a+y_1/b}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} = \frac{x_1^2/a^2-y_1^2/b^2}{\frac{1}{a^2}+\frac{1}{b^2}} = \frac{1}{\frac{1}{a^2}+\frac{1}{b^2}} = \frac{a^2b^2}{a^2+b^2},$$

since $(x_1, y_1)$ is on the curve. Thus the product of the perpendiculars is always equal to $a^2b^2/(a^2+b^2)$.

### Exercises.

35. The ordinate through a point $P$ of a hyperbola meets the asymptotes in $Q$ and $Q'$ and the hyperbola again in $P'$: show that

$$PQ\,.\,PQ'=b^2.$$

[If $PM$, $PM'$ are the perpendiculars on the asymptotes, then show that the ratios $PM/PQ$, $PM'/PQ'$ are constant. Hence, as $PM\,.\,PM'$ is constant, so also is $PQ\,.\,PQ'$. To find the constant value, let $P$ be at $A$.]

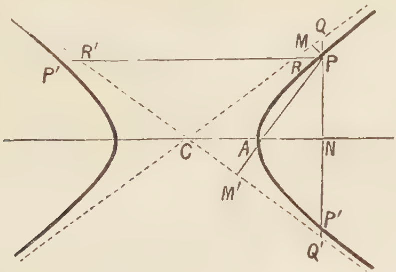



Fig. 39.

36. Show that $$QP\,.\,QP'=b^2.$$

37. If a line $PRR'P''$ parallel to the transverse axis meet the asymptotes in $R$, $R'$ and the curve again in $P''$, show that

(i.) $PR\,.\,RP''=a^2$; (ii.) $PR=R'P''$; (iii.) $PR\,.\,PR'=a^2$.

38. Show that, the further $P$ moves along the branch $AP$, the smaller do $PM$ and $PQ$ become; and that, by taking $P$ far enough away, they may be made as small as we please.

Thus we see that **the curve approaches infinitely close to its asymptote at a great distance from the centre.**

**90. The equation of a hyperbola is of the second degree whatever be the axes of coordinates, and it differs from the equation of the two asymptotes in the constant term only.**

We have obtained the equations of the hyperbola and of the asymptotes in the forms

$$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}-1=0 \text{ and } \frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=0,$$

which only differ in the constant term.

To change to any other axes, as usual we have to replace $x$ and $y$ by expressions of the form $l_1x+m_1y+n_1$ and $l_2x+m_2y+n_2$, respectively (Part I., § 35). Thus the new equations are

$$\frac{(l_1x+m_1y+n_1)^2}{\alpha^2}-\frac{(l_2x+m_2y+n_2)^2}{\beta^2}-1=0$$

and $$\frac{(l_1x+m_1y+n_1)^2}{\alpha^2}-\frac{(l_2x+m_2y+n_2)^2}{\beta^2}=0.$$

These equations are clearly the same, except for the constant term wherein the first contains the part $-1$ which is wanting in the second. This proves the result.

It must not be supposed that the difference in the constant term is always unity, for the equations are unaltered by multiplying them across by any constant, and so the terms independent of $x$ and $y$ may differ by any quantity.

**91. If, in the equation of a hyperbola, the terms of the second degree are $ax^2+2hxy+by^2$, then**

$$ab<h^2.$$

For, in the equation written above, the terms of the second degree are

$$\frac{(l_1x+m_1y)^2}{\alpha^2}-\frac{(l_2x+m_2y)^2}{\beta^2},$$

*i.e.*, the *difference* between two squares so that $ax^2+2hxy+by^2$ must have real factors, and the condition for this is

$$ab<h^2.$$

This we can easily show by comparing coefficients, for we have

$$a = \frac{l_1^2}{\alpha^2} - \frac{l_2^2}{\beta^2}, \quad b = \frac{m_1^2}{\alpha^2} - \frac{m_2^2}{\beta^2}, \quad h = \frac{l_1 m_1}{\alpha^2} - \frac{l_2 m_2}{\beta^2}.$$

Hence $$(ab - h^2) = -\left(\frac{l_1 m_2 - l_2 m_1}{\alpha\beta}\right)^2,$$

and is negative, since a square is positive.

We shall see afterwards that, when $ab < h^2$, the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ always represents a hyperbola; we have here proved the converse of this.

**Alternative proof:—**

If $$p_1x + q_1y + r_1 = 0 \quad \text{and} \quad p_2x + q_2y + r_2 = 0$$
be the asymptotes, since the product of the perpendiculars on them from a point on the curve is constant, we have

$$\frac{p_1x + q_1y + r_1}{\sqrt{p_1^2 + q_1^2}} \cdot \frac{p_2x + q_2y + r_2}{\sqrt{p_2^2 + q_2^2}} = c$$

as the equation of the curve, and this clearly only differs from that of the asymptotes in the constant term.

We leave the reader to obtain the equation of the hyperbola whose focus is $(x_1, y_1)$ and directrix $x \cos\alpha + y \sin\alpha - p = 0$, and hence to see that $ab < h^2$. (See § 70.)

**92. To find the equation of a hyperbola referred to the asymptotes as axes.**

Let $CR$, $CS$ (Fig. 40) be the asymptotes and $PM$, $PN$ the perpendiculars from a point on the curve; then we have seen that the product $PM . PN$ is constant.

Now, if $CR$ be the axis of $x$, $CS$ the axis of $y$, and $2\omega$ the angle between them, we have

$$PM = y \sin 2\omega, \quad PN = x \sin 2\omega;$$

$$\therefore \quad xy \sin^2 2\omega = \text{const.} = k^2,$$

*i.e.*, $$xy = \frac{k^2}{\sin^2 2\omega} = c^2, \text{ say.}$$

To find the value of $c^2$ in terms of $a$ and $b$, we calculate $xy$ in the simplest case, viz., when $P$ is at the vertex $A$ of the curve.

Draw $AV$ parallel to $CS$; then

$$AV . VC = c^2.$$

But, since $$\angle VAC = \angle SCA = \angle VCA,$$

$$\therefore \quad VC = VA.$$

Again $$\frac{VA}{AC} = \frac{\sin VCA}{\sin AVC} = \frac{\sin\omega}{\sin 2\omega} = \frac{1}{2\cos\omega}.$$

Thus $$c^2 = AV.VC = \frac{AC^2}{4\cos^2\omega} = \frac{a^2}{4\cos^2\omega}.$$

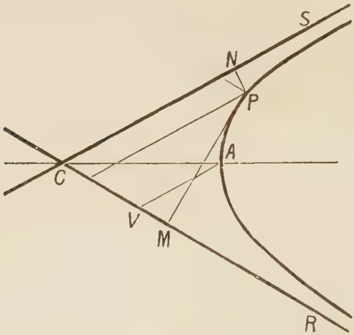



Fig. 40.

But $$\tan\omega = b/a \quad\text{or}\quad \cos^2\omega = a^2/(a^2+b^2);$$
$$\therefore\quad c^2 = \tfrac{1}{4}(a^2+b^2),$$

and the equation is $$xy = \frac{a^2+b^2}{4} \quad\text{.....................}\quad \textbf{(11)}.$$

**Alternative proof.**—The following proof is very instructive:—

To change to any other axes from the principal axes, we have to replace $x$ and $y$ in $x^2/a^2 - y^2/b^2 = 1$ by linear functions of the new coordinates, say $l_1x + m_1y + n_1$ and $l_2x + m_2y + n_2$.

If the new origin is the same as the old, $n_1 = n_2 = 0$, for both the old coordinates vanish when both the new ones do. Hence, referred to its centre as origin, the equation of the hyperbola is of the form

$$\frac{(l_1x + m_1y)^2}{a^2} - \frac{(l_2x + m_2y)^2}{b^2} = 1.$$

This is clearly of the form $Ax^2 + 2Hxy + By^2 = 1$.

Now the line $x = 0$ has to be an asymptote; therefore the quadratic $By^2 - 1 = 0$ has both roots infinite, *i.e.*, $B = 0$. Similarly, $A = 0$, and the required equation is of the form

$$2Hxy = 1 \quad\text{or}\quad xy = \text{const., as before.}$$

The value of the constant we calculate exactly as in last article. The equation is generally written

$$xy = c^2 \quad\text{.............................}\quad \textbf{(12)}$$

**Exercises.**

Find what the equations of the following hyperbolas become when they are referred to their respective asymptotes as axes of coordinates:—

39. $x^2-y^2=a^2$.   40. $2x^2-3y^2=5$.

41. $ax^2-by^2=c$.   42. $y(x-y)=2$.

43. Find the condition that $y=mx+k$ should touch the hyperbola $xy=c^2$.

44. From the result of Ex. 43 prove that every tangent to a hyperbola is inclined to an asymptote at an angle greater than the angle between the asymptotes.

## MISCELLANEOUS EXERCISES ON CHAP. V.

45. If $B$ be an extremity of the conjugate axis, prove that
$$CS^2-BC^2=a^2.$$

46. Find the equation of the hyperbola which has a focus at $(-1, 1)$, $x+y-2=0$ for corresponding directrix, and eccentricity $\sqrt{3}$.

47. Determine the latus rectum of the hyperbola in Ex. 46.

48. Show that the axis of $x$ will be an asymptote of the hyperbola
$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{if} \quad a=g=0.$$

49. The perpendicular $SQ$ from a focus on an asymptote is equal to the semi-conjugate axis, and $CQ$ is equal to half the transverse axis.

50. Show that the line $y=x+c$ will touch the hyperbola
$$x^2-2xy-y^2=1, \quad \text{provided} \quad c=\pm 1.$$

51. The asymptotes of a hyperbola are $x+y=1$ and $x-y=2$, and the sum of the squares of its axes is 5. Find its equation.

[Note that the asymptotes are at right angles.]

52. In the hyperbola show that $PN^2 : AN.A'N = BC^2 : AC^2$ where $PN$ is any ordinate.

[$PN^2=y^2$, $AN.A'N=(x-a)(x+a)$.]

53. In any ordinate $PN$ to a hyperbola a point $Q$ is taken such that $QN$ bears a constant ratio to $PN$. Show that the locus of $Q$ is a hyperbola, having exactly the same transverse axis as the original one.

54. Find the condition that the straight line $y=mx+c$ should touch the hyperbola $x^2/a^2-y^2/b^2=1$, and deduce that two real tangents can be drawn from a point $(x_1, y_1)$, if only $x_1^2/a^2-y_1^2/b^2<1$.

[Show that, if the tangent passes through $(x, y)$, there is a quadratic formed.]

55. Write down the equations of the asymptotes of the curves
$$x(x+y)=1, \quad x(x-y)=1, \quad y(x+y)=2;$$
and in general show that the asymptotes of
$$(lx+my)(l'x+m'y)=a^2$$
are
$$lx+my=0 \quad \text{and} \quad l'x+m'y=0.$$

## EXAMINATION PAPER II.

1. Define the terms *hyperbola*, *eccentricity*, *minor axis*, *transverse axis*, *latus rectum*, *asymptote*.

Prove that in the ellipse or hyperbola the minor axis is a mean proportional between the major axis and the latus rectum.

2. Trace the curve $4x^2 + 9y^2 - 4x - 6y + 1 = 0$, finding its eccentricity, the coordinates of the ends of the major and minor axes, and the length and equations of its latera recta.

3. Find, from first principles, the locus of a point that moves so that the sum of its distances from two given points is constant.

4. Interpret the equation

$$\frac{1}{r^2} - \frac{1}{a^2} = \sin^2\theta\left(\frac{1}{b^2} - \frac{1}{a^2}\right),$$

and deduce the shape of the curve.

5. Show that $$bx - y\sqrt{c^2 - a^2} = bc$$
is a tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$, and interpret the meaning of $c$.

6. Show that, if the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

represents an ellipse, then $ab - h^2$ is positive.

7. Trace the curve $4x^2 - y^2 + 4x + 4y + 13 = 0$, and find the coordinates of its foci.

8. Find the asymptotes of the hyperbola $x^2/a^2 - y^2/b^2 = 1$, and show that the angle between them is

$$(2 \sec^{-1} e).$$

9. State and prove a method for describing a hyperbola mechanically.

10. Find the equation of the hyperbola having $x + y + 1 = 0$ and $2x - y + 2 = 0$ as its asymptotes, and touching the line $x = 2$.

# CHAPTER VI.

## GENERAL EQUATION OF THE SECOND DEGREE.

93. In this chapter we shall consider and classify, to a certain extent, all those curves whose equations are of the second degree.

As usual, we write the general equation in the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

The reader will note that the equations of the curves discussed in the last three chapters are all of the second degree, and therefore all included in the general form above.

In the cases of the ellipse and hyperbola we saw further that there was a centre, and that the equation was simplest when that point was taken as origin. We shall now proceed to show that for the curve represented by the general equation there is generally such a point.

94. **If in a curve represented by an equation of the second degree all chords through the origin are bisected there, then the coefficients of $x$ and $y$ in the equation must be zero.**

The equation of any line through the origin is $y = mx$, and this meets the curve

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

in two points whose abscissæ are given by the equation

$$ax^2+2hmx^2+bm^2x^2+2gx+2fmx+c=0$$

or
$$x^2(a+2hm+bm^2)+2x(g+fm)+c=0.$$

Now, if the chord is bisected at the origin, the roots of this equation must be equal and opposite, *i.e.*, the coefficient of $x$ must be zero. (*Tut. Alg.*, II. § 163.)

Hence
$$g+fm=0.$$

But since *all* chords through the origin are bisected there, this equation must hold for all values of $m$, so that we must have $g = 0$ and $f = 0$, the required result.

**Conversely,** if $g = f = 0$, then all chords through the origin are bisected there.

For then the equation for the abscissæ, got by substituting $mx$ for $y$, has equal and opposite roots for all values of $m$.

95. **By a suitable change of origin we can always bring the equation of the curve to a form in which the coefficients of $x$ and $y$ are zero unless $ab = h^2$.**

Let us take an arbitrary new origin $(x', y')$. Then, to find the equation referred to parallel axes through $(x', y')$, we have to substitute $x + x'$ for $x$ and $y + y'$ for $y$ in the original equation. Hence the new equation is

$$a(x+x')^2 + 2h(x+x')(y+y') + b(y+y')^2 \\ + 2g(x+x') + 2f(y+y') + c = 0$$

or $$ax^2 + 2hxy + by^2 + 2x(ax' + hy' + g) + 2y(hx' + by' + f) \\ + ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0.$$

Now in this equation the coefficients of $x$ and $y$ are zero if $ax' + hy' + g = 0$ and $hx' + by' + f = 0$ ...... (A), or, on solving by the method of cross multiplication, we must have

$$\frac{x'}{hf - bg} = \frac{y'}{gh - af} = \frac{1}{ab - h^2}, \quad (\textit{Tut. Alg.}\ \text{II.}, \S\, 68)$$

*i.e.*, $$x' = \frac{hf - bg}{ab - h^2}, \quad y' = \frac{gh - af}{ab - h^2}.$$

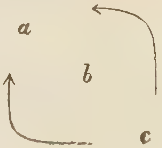
Hence, unless $ab = h^2$, we can choose finite values of $x'$ and $y'$ so that the new equation has no terms in $x$ and $y$.

The two equations (A) can be easily remembered if we adopt the following device:—

Write down the letters $a$, $b$, $c$ as the diagonal of a square containing three dots in each side, thus:

$$\begin{array}{ccc} a & \cdot & \cdot \\ \cdot & b & \cdot \\ \cdot & \cdot & c \end{array}$$

Then fill up the vacant spaces with the letters $f$, $g$, $h$ as indicated by the arrow-heads.



We thus get
$$\begin{matrix} a & h & g \\ h & b & f \\ g & f & c \end{matrix}$$

The letters in the first two lines are respectively the coefficients of $x'$, $y'$, and the absolute terms in the equations of (A).

Combining the last three articles, we see that,

**96. Unless $ab = h^2$, there exists a point connected with a curve of the second degree such that all chords through it are bisected there.**

For we have seen that, if we take the point $(x', y')$ of § 95 for origin, there are no terms in $x$ and $y$ in the equation, and then the previous articles show that all chords through the new origin are bisected there.

This point is called the **centre** of the curve, and every line through it is called a **diameter.**

NOTE.—The student will easily see that all this agrees with what was said as to the centre in Chaps. IV. and V.

COR. The coordinates of the centre of the curve are given by the equations

$$ax' + hy' + g = 0, \quad hx' + by' + f = 0 \;\ldots\ldots\; (1);$$

and, when the centre is taken as new origin, the equation becomes

$$ax^2 + 2hxy + by^2 + ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0.$$

*Example.—Write down the equations for the centre of the curve*

$$3x^2 - 2xy + y^2 - x - 3y + 1 = 0,$$

*and find the coordinates of the centre.*

Here $a = 3$, $h = -1$, $b = 1$, $g = -\frac{1}{2}$, $f = -\frac{3}{2}$, $c = 1$.

Therefore the equations for the centre

$$ax + hy + g = 0, \quad hx + by + f = 0$$

become $$3x - y - \tfrac{1}{2} = 0, \quad -x + y - \tfrac{3}{2} = 0,$$

whence $$x = 1, \quad y = \tfrac{5}{2}.$$

**Exercises.**

Write down the equations for the centre, and find the centre, of each of the following curves:—

1. $x^2 + 2xy + 2y^2 + 8x + 3 = 0$.
2. $2x^2 - y^2 - 4x + 2y = 7$.
3. $x^2 - 2xy + 3x + y + 3 = 0$.
4. $x^2 + x + y + 1 = 0$.

**97. Equation of curve referred to the centre as origin.**

RULE.—*The equation referred to the centre as origin is found by substituting half the coordinates of the centre in the terms of the first degree in the original equation.*

We have seen that the equation required is

$$ax^2+2hxy+by^2+ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0.$$

Now the new absolute term is

$$\begin{aligned} ax'^2+2hx'y'+by'^2+2gx'+2fy'+c \\ &= x'(ax'+hy'+g)+y'(hx'+by'+f)+gx'+fy'+c \\ &= 2g\left(\frac{x'}{2}\right)+2f\left(\frac{y'}{2}\right)+c, \end{aligned}$$

since $\quad ax'+hy'+g=0 \quad$ and $\quad hx'+by'+f=0.$

But, since the terms of the first degree in the equation are $2gx+2fy+c$, this proves the rule enunciated above.

This rule is very important, as it shortens the work in practice—the student should remember it.

*Example.—Find the coordinates of the centre of the curve*

$$3x^2+xy+y^2+4x+y+1=0,$$

*and deduce the equation referred to the centre as origin.*

Here we have $a=3$, $h=\frac{1}{2}$, $b=1$, $g=2$, $f=\frac{1}{2}$, $c=1$.

The equations for the centre are

$$ax'+hy'+g=0 \quad \text{and} \quad hx'+by'+f=0,$$

or

$$3x'+\tfrac{1}{2}y'+2=0 \quad \text{and} \quad \tfrac{1}{2}x'+y'+\tfrac{1}{2}=0.$$

Solving, we find $\quad x'=-\frac{7}{11}, \quad y'=-\frac{2}{11}$

as the coordinates of the centre.

Substituting half these coordinates in the terms of the first degree, viz., $4x+y$, the equation becomes

$$3x^2+xy+y^2+4\left(-\tfrac{7}{22}\right)+\left(-\tfrac{1}{11}\right)+1=0 \quad \text{or} \quad 3x^2+xy+y^2-\tfrac{4}{11}=0.$$

**Exercises.**

Find the centres of the following curves and their equations when the centres are taken as origin:—

5. $3x^2+2xy-y^2+2x+4y-1=0.$ 6. $4x^2+y^2+x+y+1=0.$

7. $xy+2y^2+4x+3y+17=0.$

8. In the general case, show that the straight lines $ax+hy+g=0$ $hx+by+f=0$ are diameters of the curve.

98. **New absolute term in terms of the original coefficients.**

In the general case, the new absolute term is

$$gx'+fy'+c,$$

and this, written in the form

$$2g\left(\frac{x'}{2}\right)+2f\left(\frac{y'}{2}\right)+c,$$

the student should always use in calculation; we may, as a matter of theory, work out the value in terms of $a, b, c, f, g, h$. In fact, we have

$$gx'+fy'+c=\frac{g(hf-bg)}{ab-h^2}+\frac{f(gh-af)}{ab-h^2}+c,$$

on putting in for $x'$ and $y'$ their values (§ 95)

$$=\frac{fgh-bg^2+fgh-af^2+abc-ch^2}{ab-h^2}$$

$$=\frac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}.$$

Hence the equation referred to the centre as origin is

$$\boldsymbol{ax^2+2hxy+by^2+\frac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}=0} \qquad \textbf{(2)}.$$

**Caution.**—As we have said, this formula for the new absolute term should not be used for calculation. The rule previously given is much more suitable, as in practice we nearly always require in addition the coordinates of the centre.

Cor. The new constant term is zero, if

$$abc+2fgh-af^2-bg^2-ch^2=0,$$

that is, only when the curve of the second degree represents two straight lines (Part I., § 32). In this case the equation referred to the new origin is

$$ax^2+2hxy+by^2=0,$$

which represents two lines meeting in the origin.

Thus, when the general equation represents two straight lines, their point of intersection is the point obtained by finding the centre of the curve in the usual way.

It is, in fact, clear from Geometry that this must be the case; for, if $O$ be the point of intersection and $POP'$ a line through $O$ such that $OP=OP'$, then, whenever $P$ is on one of the lines, so also is $P'$, and this is precisely the condition that the centre of the curve satisfies.

99. **An equation of the second degree can, by a suitable change of origin, be reduced to the form**

$$Ax^2+2Hxy+By^2=1,$$

**provided that, in the original equation,**

$$ab \neq h^2 \quad \text{and} \quad abc+2fgh-af^2-bg^2-ch^2 \neq 0.$$

For we have seen that the terms of the first degree in $x$ and $y$ can be got rid of by taking the centre as origin, and the equation then becomes

$$ax^2+2hxy+by^2=c',$$

where $c'$* is $\quad -\dfrac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}$. (§ 98)

Hence, dividing across by $c'$, and putting

$$a/c'=A,\ h/c'=H,\ b/c'=B,$$

we obtain $\quad Ax^2+2Hxy+By^2=1$ ............... **(3).**

100. **By turning the axes round through a suitable angle $\theta$, we can reduce the equation**

$$Ax^2+2Hxy+By^2=1$$

**to the form** $\quad \alpha x^2+\beta y^2=1.$

To turn the axes round through an angle $\theta$, we have (Part I., § 33) to substitute

$x\cos\theta-y\sin\theta$ for $x$ and $x\sin\theta+y\cos\theta$ for $y$.

Hence the new equation becomes

$$A(x\cos\theta-y\sin\theta)^2+2H(x\cos\theta-y\sin\theta)(x\sin\theta+y\cos\theta)$$
$$+B(x\sin\theta+y\cos\theta)^2=1,$$

*i.e.*, $\quad x^2(A\cos^2\theta+2H\sin\theta\cos\theta+B\sin^2\theta)$

$$-2xy\{(A-B)\sin\theta\cos\theta-H(\cos^2\theta-\sin^2\theta)\}$$
$$+y^2\{A\sin^2\theta-2H\sin\theta\cos\theta+B\cos^2\theta\}=1.$$

Now in this equation the coefficient of $xy$ is zero if only

$$(A-B)\sin\theta\cos\theta=H(\cos^2\theta-\sin^2\theta),$$

*i.e.*, if $\quad \dfrac{2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}=\dfrac{2H}{A-B};$

$$\therefore \quad \tan 2\theta=\frac{2H}{A-B} \text{ ..................... } \mathbf{(4).}$$

---

* Its actual value, so long as it is not zero, does not affect the argument.

Now an angle less than 180° can be determined whose tangent is any real quantity, and hence this equation shows us through what angle the axes must be turned to get rid of the term in $xy$.

Hence the equation of the second degree can be reduced to the form $$\alpha x^2+\beta y^2 = 1,$$
where $$\alpha = A\cos^2\theta+2H\sin\theta\cos\theta+B\sin^2\theta$$
and $$\beta = A\sin^2\theta-2H\sin\theta\cos\theta+B\cos^2\theta.$$

**Exercises.**

9. Reduce the equations of Exx. 5–7 to the form $Ax^2+2Hxy+By^2=1$.

10. **Unless $ab=h^2$, the general equation of the second degree represents an ellipse or a hyperbola.**

### 101. Reduction of the equation in actual practice.

Although the quantities $\alpha$ and $\beta$ *can* be found by actually turning the axes round as we have done in § 100, such a process is tedious, and accordingly in actual practice another method is adopted, which we proceed to explain. On comparing it with the last, the reader will see at once that it assumes that the curve represented is a conic, and therefore cannot replace the method of § 100 in the proof that the general equation can be reduced to the form
$$\alpha x^2+\beta y^2 = 1.$$
The method depends on the following lemma.

### 102. A central conic is met by a concentric circle in four points which lie by twos on two lines through the centre equally inclined to the axes of the conic.

This follows at once, since both kinds of central conics are symmetrical with respect to their axes, but we can give a formal proof, as follows. Suppose the central conic is $\alpha x^2+\beta y^2=1$, and the circle is $x^2+y^2=r^2$; then the lines joining the centre to the common points are
$$\alpha x^2+\beta y^2=\frac{x^2+y^2}{r^2},$$
for this represents a pair of lines through the origin, and both sides are equal to unity at the points where the curves meet.

Transposing, we get
$$x^2\left(\alpha-\frac{1}{r^2}\right)=y^2\left(\frac{1}{r^2}-\beta\right),$$
and this equation plainly represents a pair of lines equally inclined to their axes.

Cor. *The two lines coincide only when $r$ is equal to a semi-axis of the conic, and then they coincide in the corresponding axis.*

**103. To find the equations and lengths of the semi-axes of the conic whose equation is**

$$Ax^2+2Hxy+By^2=1.$$

[**Caution.**—Note carefully that the right-hand side is unity.]

We have seen that such a conic and the circle

$$x^2+y^2=r^2$$

have a pair of common chords equally inclined to the axes of the conic, and these chords coincide in one of the axes if $r$ is the length of the axis.

But the equation of the pair of lines joining the centre to the common points is found by making the first equation homogeneous by means of the second, and is therefore

$$Ax^2+2Hxy+By^2=\frac{x^2+y^2}{r^2},$$

On transposing we get

$$x^2\left(A-\frac{1}{r^2}\right)+2Hxy+y^2\left(B-\frac{1}{r^2}\right)=0.$$

Now this pair of lines will be a coincident pair only if the expression of the left-hand side is a perfect square, *i.e.*, if

$$H^2=\left(A-\frac{1}{r^2}\right)\left(B-\frac{1}{r^2}\right) \ldots\ldots\ldots\ldots\ldots \textbf{(5)}.$$

Thus $r$ is given by the equation

$$H^2=AB-(A+B)\frac{1}{r^2}+\frac{1}{r^4}$$

or

$$\frac{1}{r^4}-\frac{1}{r^2}(A+B)+AB-H^2=0 \ldots\ldots\ldots \textbf{(5)}.$$

By solving this as a quadratic in $\frac{1}{r^2}$ we find two roots which are the squares of the reciprocals of the semi-axes. Suppose $\frac{1}{r_1^2}$ and $\frac{1}{r_2^2}$ are the roots; then $r_1$ and $r_2$ are the

semi-axes, and now

$$\left(A - \frac{1}{r_1^2}\right)x^2 + 2Hxy + \left(B - \frac{1}{r_1^2}\right)y^2 = 0$$

is the square of the equation of one axis, and

$$\left(A - \frac{1}{r_2^2}\right)x^2 + 2Hxy + \left(B - \frac{1}{r_2^2}\right)y^2 = 0$$

is the square of the equation of the other.

The first of these equations, on multiplying across by

$$\left(A - \frac{1}{r_1^2}\right),$$

becomes

$$\left(A - \frac{1}{r_1^2}\right)^2 x^2 + 2H\left(A - \frac{1}{r_1^2}\right)xy + \left(A - \frac{1}{r_1^2}\right)\left(B - \frac{1}{r_1^2}\right)y^2 = 0$$

or

$$\left(A - \frac{1}{r_1^2}\right)^2 x^2 + 2H\left(A - \frac{1}{r_1^2}\right)xy + H^2y^2 = 0,$$

since
$$\left(A - \frac{1}{r_1^2}\right)\left(B - \frac{1}{r_1^2}\right) = H^2.$$

Hence the equation of the semi-axis whose length is $r_1$ is

$$\boldsymbol{\left(A - \frac{1}{r_1^2}\right)x + Hy = 0} \quad \ldots\ldots\ldots\ldots\ldots \textbf{(6)},$$

and the equation of the other is, in like manner,

$$\boldsymbol{\left(A - \frac{1}{r_2^2}\right)x + Hy = 0}.$$

**Caution.**—The student should remember that the lemma (§ 102) preceding this investigation is a part of the whole proof, and hence he should give the lemma in proving the proposition. Further—and this is very important—he should carefully bear in mind that the equation we deal with has unity for its absolute term, and therefore in commencing an example it is essential to divide across in such a way that the absolute term becomes unity. If this is not done, we do not get the semi-axes of the curve in question, but those of a curve of the same shape while not of the same size. Then finally, by way of checking his work, he should verify that the axes he finds are actually at right angles.

*Alternative method for finding the lengths of the semiaxes.*—We can find the lengths of the semiaxes by making use of the invariants (Part I., § 37).

Suppose that $\alpha$, $\beta$ are the semiaxes of

$$ax^2+2hxy+by^2=1;$$

then the equation referred to the axes is

$$\frac{x'^2}{\alpha^2}+\frac{y'^2}{\beta^2}=1$$

Therefore, by some change of rectangular axes,

$$ax^2+2hxy+by^2 \quad \text{becomes} \quad \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}.$$

Hence $\quad a+b=\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$ and $ab-h^2=\dfrac{1}{\alpha^2}\cdot\dfrac{1}{\beta^2}$.

Now, by the theory of quadratic equations, $\dfrac{1}{\alpha^2}$ and $\dfrac{1}{\beta^2}$ are the roots of the quadratic in $t$

$$t^2-t\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)+\frac{1}{\alpha^2\beta^2}=0.$$

Therefore $\dfrac{1}{\alpha^2}$ and $\dfrac{1}{\beta^2}$ are the roots of

$$t^2-t(a+b)+ab-h^2=0,$$

the same result as that in (5).

Note that this method only determines the lengths and not the equations of the semiaxes. But it applies equally well when the axes are oblique, and, in fact, if $\omega$ be the angle between them, we have

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{1}{\alpha^2}+\frac{1}{\beta^2},$$

$$\frac{ab-h^2}{\sin^2\omega}=\frac{1}{\alpha^2}\cdot\frac{1}{\beta^2},$$

since the $x'$, $y'$ axes are rectangular.

Thus, in this case, $\dfrac{1}{\alpha^2}$ and $\dfrac{1}{\beta^2}$ are the roots of

$$t^2-t\frac{a+b-2h\cos\omega}{\sin^2\omega}+\frac{ab-h^2}{\sin^2\omega}=0$$

or $$t^2\sin^2\omega-t(a+b-2h\cos\omega)+ab-h^2=0.$$

104. *Example* (i.). *Find the equations and lengths of the semi-axes of the conic*
$$5x^2 + 4xy + 2y^2 = 1.$$

Here, if $r$ be a semi-axis, the pair of lines
$$5x^2 + 4xy + 2y^2 = \frac{x^2 + y^2}{r^2}$$
coincide in that axis. Thus $r$ is given by
$$\left(5 - \frac{1}{r^2}\right)\left(2 - \frac{1}{r^2}\right) = 4 \quad \text{or} \quad \frac{1}{r^4} - \frac{7}{r^2} + 6 = 0,$$
*i.e.*,
$$\left(\frac{1}{r^2} - 1\right)\left(\frac{1}{r^2} - 6\right) = 0\,;$$
$$\therefore \quad r^2 = 1 \text{ or } \tfrac{1}{6}.$$

The equation of the axis is
$$\left(A - \frac{1}{r^2}\right)x + Hy = 0 \quad \text{or} \quad \left(5 - \frac{1}{r^2}\right)x + 2y = 0.$$

If $r^2 = 1$, this gives $\quad (5-1)x + 2y = 0$, *i.e.*, $2x + y = 0$.

If $r^2 = \frac{1}{6}$, the equation becomes $(5-6)x + 2y, =$ *i.e.*, $2y - x = 0$.

Thus the lengths of the semi-axes are 1 and $\dfrac{1}{\sqrt{6}}$, and their equations are
$$2x + y = 0 \quad \text{and} \quad 2y - x = 0, \quad \text{respectively,}$$
and the fact that the two are at right angles confirms the accuracy of the work. [This test should always be applied, at least mentally.]

*Example* (ii.). *To find the equation of the above conic referred to its axes as axes of coordinates.*

Since the lengths of the axes are 1 and $\dfrac{1}{\sqrt{6}}$, the equation required is
$$\frac{x^2}{1} + \frac{y^2}{\frac{1}{6}} = 1 \quad \text{or} \quad x^2 + 6y^2 = 1,$$
where $OX$ is the major axis and $OY$ the minor axis.

*Example* (iii.). *Find the equations and lengths of the semi-axes of the curve*
$$7x^2 + 6xy - y^2 = 4.$$

We must begin by dividing across by 4, and then the equation is
$$\tfrac{7}{4}x^2 + \tfrac{6}{4}xy - \tfrac{1}{4}y^2 = 1.$$

The equation giving the lengths is
$$\left(a - \frac{1}{r^2}\right)\left(b - \frac{1}{r^2}\right) = h^2 \quad \text{or} \quad \left(\frac{7}{4} - \frac{1}{r^2}\right)\left(-\frac{1}{4} - \frac{1}{r^2}\right) = \left(\frac{3}{4}\right)^2,$$
$$\therefore \quad \frac{1}{r^4} - \frac{1}{r^2}\cdot\frac{6}{4} - 1 = 0\,;$$

$$\therefore \quad \frac{1}{r^2} = 2 \text{ or } -\tfrac{1}{2}; \qquad \therefore \quad r = \frac{1}{\sqrt{2}} \text{ or } \sqrt{-2}.$$

Thus, one semi-axis being imaginary, the curve is a hyperbola.
The equation of an axis is

$$\left(A - \frac{1}{r^2}\right) x + Hy = 0.$$

If $\dfrac{1}{r^2} = 2$, this gives $\left(\tfrac{7}{4} - 2\right) x + \tfrac{3}{4} y = 0$ or $x - 3y = 0$.

If $\dfrac{1}{r^2} = -\tfrac{1}{2}$, it gives $\left(\tfrac{7}{4} + \tfrac{1}{2}\right) x + \tfrac{3}{4} y = 0$ or $3x + y = 0$.

As usual, we note in confirmation that the axes obtained are at right angles.

*Example* (iv.). *Find the equation of the above curve referred to its semi-axes.*

Since the values of $r^2$ are $\frac{1}{2}$ and $-2$, the equation required is

$$\frac{x^2}{\frac{1}{2}} + \frac{y^2}{-2} = 1 \quad \text{or} \quad 2x^2 - \frac{y^2}{2} = 1,$$

*i.e.*

$$4x^2 - y^2 = 2,$$

the transverse axis being axis of $x$.

*Example* (v.). *To verify that the axes obtained by the general method are at right angles.*

The axes are

$$\left(A - \frac{1}{r_1^2}\right) x + Hy = 0 \quad \text{and} \quad \left(A - \frac{1}{r_2^2}\right) x + Hy = 0,$$

where $\dfrac{1}{r_1^2}$ and $\dfrac{1}{r_2^2}$ are the roots of the equation in $t$,

$$t^2 - t(A + B) + (AB - H^2) = 0.$$

The two lines are at right angles if only

$$\left(A - \frac{1}{r_1^2}\right)\left(A - \frac{1}{r_2^2}\right) + H^2 = 0,$$

*i.e.*, if

$$A^2 - A\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right) + \frac{1}{r_1^2 r_2^2} + H^2 = 0.$$

Now

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} = A + B \quad \text{and} \quad \frac{1}{r_1^2 r_2^2} = AB - H^2,$$

by the theory of quadratics. Consequently the condition required is

$$A^2 - A(A + B) + AB - H^2 + H^2 = 0,$$

which is obviously true.

Thus the lines are at right angles, as they should be.

**Exercises.**

Find the equations and lengths of the semi-axes of the following conics, and write down their equations referred to those lines as axes of coordinates :—

11. $3x^2+2xy+3y^2 = 1$.

12. $7x^2+4xy+4y^2 = 1$.

13. $x^2+xy+y^2 = 3$.

14. $11x^2-72xy-54y^2 = 1$.

15. By applying the test for the sign of $ab-h^2$, determine whether the above curves are ellipses or hyperbolas.

16. Show that the lengths of the semi-axes of the conic

$$(m^2+a)\,x^2+2mnxy+(n^2+a)\,y^2 = 1$$

are $\dfrac{1}{\sqrt{m^2+n^2+a}}$ and $\dfrac{1}{\sqrt{a}}$, and that their equations are

$$nx-my = 0, \quad mx+ny = 0.$$

17. The axes of the conic $Ax^2+2Hxy+By^2 = 1$ are given by

$$\left(A-\frac{1}{r_1^2}\right)x+Hy = 0, \quad \left(A-\frac{1}{r_2^2}\right)x+Hy = 0,$$

where $\dfrac{1}{r_1^2}$, $\dfrac{1}{r_2^2}$ are the roots of the quadratic

$$t^2-t\,(A+B)+AB-H^2 = 0.$$

Deduce their joint equation, viz., $H\,(x^2-y^2)-(A-B)\,xy = 0$, and show that this follows at once from the fact that the axes bisect the angles between the asymptotes.

105. *To show that the roots of the equation in* $\dfrac{1}{r^2}$ *are always real.*

The roots of the equation

$$\frac{1}{r^4}-\frac{1}{r^2}(A+B)+AB-H^2 = 0$$

will be real if $\qquad (A+B)^2-4\,(AB-H^2) \not< 0$, (*Tut. Alg.*, II., § 159)

*i.e.*, if $\qquad (A-B)^2+4H^2 \not< 0$.

Now the expression on the left, being the sum of two squares, cannot be negative, and hence the result follows.

Cor. If the semi-axes are equal, we must have

$$A-B = 0 \quad \text{and} \quad H = 0;$$

for then the expression $(A-B)^2+4H^2$ must be zero.

In this case the equation, of course, represents a circle, for the ellipse becomes a circle when its axes are equal, and, as we knew before, the equation of a circle is of the form here indicated.

The reader might have expected only one condition, for only one condition is necessary in general to ensure a quadratic having equal roots. There are two, because the expression which has to vanish is the sum of the squares of two real quantities.

**106. The equation $Ax^2+2Hxy+By^2=1$ represents an ellipse if $AB > H^2$, and a hyperbola if $AB < H^2$.**

The quadratic for $\frac{1}{r^2}$ is

$$\frac{1}{r^4}-\frac{1}{r^2}(A+B)+AB-H^2=0,$$

and we have seen that the roots are real. If they are of opposite signs, the conic is a hyperbola; if of the same sign, it is an ellipse (see the forms in §§ 55, 73).

But they are of the same sign or not according as $AB-H^2$ is positive or negative.

Hence, if $AB-H^2$ is positive, the equation denotes an ellipse (a real ellipse if the two roots in $1/r^2$ are positive, an imaginary ellipse if they are negative); whereas, if $AB-H^2$ is negative, the equation denotes a hyperbola.

Cor. If $AB = H^2$, the equation denotes a pair of parallel straight lines, for the portion on the left-hand side is a perfect square, say $(\alpha x+\beta y)^2$, and hence $\alpha x+\beta y = \pm 1$, denoting a pair of parallel lines.
But it must be noticed that, in the discussion of the *general* equation, the case of $ab = h^2$ has been left aside throughout.

**107. Supposing the equation of the second degree represents a hyperbola, to find the equation of its asymptotes.**

We have seen that the joint equation of the two asymptotes differs from the equation of the curve only in the constant term (§ 86); hence we have the following

Rule.—*To find the equation of the asymptotes, replace the absolute term in the given equation by an unknown quantity $\rho$, and then determine $\rho$ so that the new equation represents two straight lines.*

*Example.—Find the equation of the asymptotes of the conic*

$$x^2-4xy+3y^2+2x-4y+3 = 0.$$

We have here to find $\rho$ so that

$$x^2-4xy+3y^2+2x-4y+\rho = 0$$

represents two straight lines. The condition is

$$3\rho+8-4-3-4\rho = 0, \qquad \text{(Part I., § 32)}$$

so that $\rho = 1$ and the asymptotes are given by

$$x^2 - 4xy + 3y^2 + 2x - 4y + 1 = 0 \quad \text{or} \quad (x-3y+1)(x-y+1) = 0,$$

so that they are the two lines

$$x - 3y + 1 = 0 \quad \text{and} \quad x - y + 1 = 0.$$

**Exercises.**

Find the equation of the asymptotes of the following curves:—

18. $x^2 + 2xy - 2y^2 + x + y = 0$.

19. $4x^2 + 14xy + 7y^2 + 11x + 9y + 7 = 0$. 20. $3x^2 + xy - y^2 + 2y = 0$.

108 **Two conics whose equations differ only in the constant term have the same asymptotes.**

For, in finding the equation of the asymptotes, we use all the coefficients but the absolute term, and not that; hence the equation found depends only on the first five coefficients, and not on the absolute term

109. **Asymptotes of an ellipse.**

It will be noticed that we obtain an equation for the asymptotes whether the curve is an ellipse or a hyperbola; the difference between the cases lies in the fact that, while for the hyperbola the equation determined for the asymptotes must split up into real factors, that for the asymptotes of the ellipse splits up only into imaginary factors, so that *the asymptotes of an ellipse are imaginary.*

110. **To find the asymptotes of the conic given by the general equation.**

Following the rule, we have to replace the absolute term by a quantity so chosen that the new expression is the product of two factors. Let us then replace $c$ by $c + c'$ where $c'$ has to be determined.

Then, since $ax^2 + 2hxy + by^2 + 2gx + 2fy + c + c' = 0$

epresents two lines, we have

$$ab(c + c') + 2fgh - af^2 - bg^2 - (c + c')h^2 = 0.$$

Hence we have $c' = -\dfrac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2}$,

and hence the equation of the asymptotes is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2} = 0$$

Cor. I. $c' = 0$ if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$,

*i.e.*, if the original equation represents two straight lines.

$c' = \infty$ if $ab = h^2$,

*i.e.*, if the equation represents a parabola, and here, as we have seen, there are no asymptotes at a finite distance (§ 49).

Cor. II. The asymptotes of $ax^2+2hxy+by^2+2gx+2fy+c=0$ are parallel to the pair of lines $ax^2+2hxy+by^2=0$.

For their equation is of the form

$$ax^2+2hxy+by^2+2gx+2fy+c+c'=0,$$

and this pair of lines is parallel to the pair represented by

$$ax^2+2hxy+by^2=0. \qquad \text{(Part I., § 32.)}$$

*To find the asymptotes, then, we draw lines through the centre parallel to the lines* $ax^2+2hxy+by^2=0$.

Cor. III. The equation represents an ellipse or hyperbola according as $ab >$ or $< h^2$.

For $ax^2+2hxy+by^2$ has imaginary or real factors according as $ab >$ or $< h^2$., *i.e* the asymptotes are imaginary or real according as $ab >$ or $< h^2$. (*Cf.* § 107.)

### 111. Condition for a rectangular hyperbola.

In this case the asymptotes are at right angles; thus the lines

$$ax^2+2hxy+by^2=0$$

are at right angles, and accordingly the condition is $a+b=0$.

(Part I., § 29.)

Thus the equation of the second degree represents a rectangular hyperbola when the coefficients of $x^2$ and $y^2$ are numerically equal but opposite in sign.

### 112. To find the asymptotes by inspection.

The asymptotes of a conic may sometimes be found by inspection. Thus, if the equation be $(x+3y)(x+2y+1)=4$, the asymptotes are clearly $x+3y=0$, $x+2y+1=0$, for their joint equation differs from that of the curve only in the constant term.

Again, when the terms in $x^2$ and $y^2$ are wanting, the same method always applies. Thus, to find the asymptotes of

$$xy+2x-y+4=0$$

we write it

$$(x-1)(y+2)+6=0,$$

and the asymptotes are plainly $x-1=0$ and $y+2=0$.

#### Exercises.

Find by inspection the asymptotes of

21. $x(x+y)=1$. 22. $y(y-x)=1$. 23. $xy+2x+3y=0$.

24. $x^2+xy+x+y+6=0$. 25. $xy+2x-y=0$.

26. $x(2x+y)=x+1$. 27. $(x-y)(3x+2y)=6x+4y+5$.

28. Deduce from the results of Exx. 21–27 the centres of the respective curves.

## MISCELLANEOUS EXERCISES ON CHAP. VI.

29. Find the coordinates of the centre of the conic whose equation

$$5x^2+6xy-5y^2-22x+18y-7=0.$$

30. Transform the equation of the conic

$$25x^2-36xy+40y^2+10x-28y-47=0$$

to parallel axes through its centre.

31. Transform the equation of the conic

$$57x^2-150xy-23y^2=34$$

to its principal diameters as axes of coordinates.

32. Find the equation of the diameters of the curve

$$Ax^2+2Hxy+By^2=1$$

passing through its points of intersection with the concentric circle

$$x^2+2xy\cos\omega+y^2=r^2.$$

33. Prove that the asymptotes of

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

are given by the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c$$

where $x'$, $y'$ are the coordinates of the centre.

34. Show that, if $ax^2+2hxy+by^2=1$ and $a'x^2+2h'xy+b'y^2=1$ represent the same conic, referred to two different sets of rectangular axes, then $a+b=a'+b'$ and $ab-h^2=a'b'-h'^2$.

Find the equations of the hyperbolas passing through the point (1, 2) and having the following lines as asymptotes :—

35. $3x-y+1=0,\ x+y=0.$

36. $x=0,\ y=3.$

37. $x+2y+7=0,\ 3x-y=4.$

38. The asymptotes of a hyperbola being given by

$$2x^2-5xy-3y^2=0,$$

find the equation of the axes.

39. Find the joint equation of the axes of a hyperbola which has the straight lines $Ax^2+2Hxy+By^2=0$ for asymptotes.

40. Find the equation of the hyperbola having the same asymptotes as $2x^2+xy-y^2-3x+3y=9$ and passing through the origin.

# CHAPTER VII.

## TRACING OF ELLIPSES.

**113.** We shall now work out a few examples of tracing an ellipse when its equation is given in the general form.

On account of the simplicity of the form of the curve, there is no difficulty in obtaining a very good idea of the position of the curve when we know its semi-axes in magnitude and direction, accordingly, we direct our attention to finding the semi-axes, and then, by way of confirmation, we obtain a few points on the curve. All the necessary processes have been explained in the last chapter, viz.,

(i.) we find the centre of the curve and the equation when that point is taken as origin;

(ii.) having done this, we proceed to find the lengths and equations of the semi-axes.

**114.** *Example* (i.). *Trace the curve*

$$36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0.$$

[NOTE.—The following must not be regarded as a model solution, as the portions given in square brackets form no radical part of the proof. They are hints which should always be applied, at least mentally, to test the accuracy of the work as the proof is evolved.]

(*a*) Here $ab - h^2 = 36 \times 29 - 12^2 =$ a positive quantity.

[*It is clearly not necessary to determine the* actual *value of* $ab - h^2$.]

Therefore the curve is an ellipse (§ 106.)

(*b*) The equations giving the coordinates of the centre are

$$36x + 12y - 36 = 0, \quad 12x + 29y + 63 = 0,$$

whence $\quad x = 2, \quad y = -3.$

[*After obtaining the value of x and y as above, substitute in the equations to confirm your solution.*]


Fig. 41.

(c) Substituting half the coordinates of the centre in the terms of the first degree (see § 97), we get for the equation referred to the

centre $$36x^2+24xy+29y^2-72\,(1)+126\,(-\tfrac{3}{2})+81=0$$

or $$36x^2+24xy+29y^2=180$$

or $$\tfrac{1}{5}x^2+\tfrac{2}{15}xy+\tfrac{29}{180}y^2=1. \qquad \text{(See Caution, p. 96)}$$

(d) The semi-axes are given by

$$\left(A-\frac{1}{r^2}\right)\left(B-\frac{1}{r^2}\right)=H^2 \qquad (\S\,103)$$

or $$\left(\frac{1}{5}-\frac{1}{r^2}\right)\left(\frac{29}{180}-\frac{1}{r^2}\right)=\left(\frac{1}{15}\right)^2$$

or $$\frac{1}{r^4}-\tfrac{13}{36}\cdot\frac{1}{r^2}+\tfrac{1}{36}=0,$$

whence $$r^2=4 \text{ or } 9 \quad\text{and}\quad r=2 \text{ or } 3.$$

Equation of curve referred to its principal axes as axes of coordinates

is therefore $$\frac{x^2}{9}+\frac{y^2}{4}=1.$$

(e) The equation of the major or minor axis is

$$\left(A-\frac{1}{r^2}\right)x+Hy=0.$$

When $r^2=9$ (*major* axis), this is

$$(\tfrac{1}{5}-\tfrac{1}{9})\,x+\tfrac{1}{15}y=0 \quad\text{or}\quad 4x+3y=0;$$

when $r^2=4$ (*minor* axis), this is

$$(\tfrac{1}{5}-\tfrac{1}{4})\,x+\tfrac{1}{15}y=0 \quad\text{or}\quad 3x-4y=0.$$

[*At this stage, confirm your work by noting that the two lines found for axes are at right angles* (Part I., § 19).]

[*The easiest method of drawing the axes at this stage is as follows. For major axis in this case, put* $x=3$ (*because coefficient of* $y$ *is* 3), *and then* $y=-4$; *then plot this point P relative to the new axes, and draw the line CP through centre of curve and this point. For minor axis, put* $x=4$ (*because coefficient of* $y=4$), *and then* $y=3$. *Plot this point* (*Q in figure*), *and draw line as before.*]

(f) Marking off lengths 3 and 2 along the lines $4x+3y=0$ and $3x-4y=0$ respectively in both directions, we get the ends of the major and minor axes, and the curve can now be plotted.

[*Before actually drawing the curve, it is best to confirm the work by finding the points in which the curve cuts the* original *axes if it does so, and, if not, the points in which it cuts some other suitable lines. This we give in* (g).]

(*g*) The curve cuts the original axis of $y$ ($x = 0$) where

$$29y^2 + 126y + 81 = 0$$

or
$$y = \frac{-63 \pm \sqrt{(63^2 - 29 \times 81)}}{29} = \frac{-63 \pm 40}{29} \quad \text{approx.}$$

[*Note the saving of work by taking an approximate value at this stage.*]

$$\therefore \quad y = -\cdot 8 \quad \text{or} \quad -3\cdot 6, \quad \text{approx.}$$

These are represented by $N$ and $N'$ in diagram.

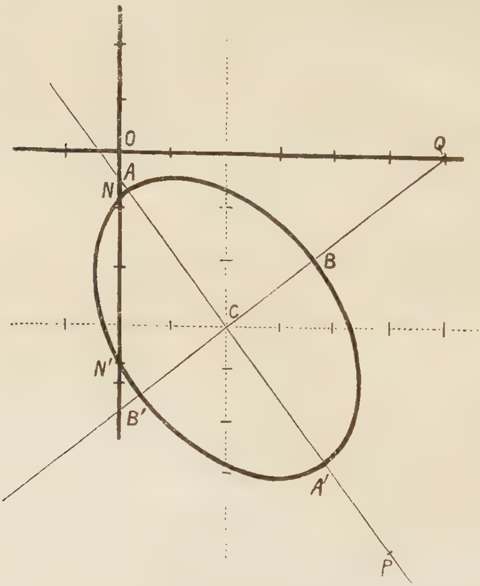



Fig. 42.

Similarly, the curve cuts the original axis of $x$ where

$$36x^2 - 72x + 81 = 0,$$

which gives *imaginary* values, *i.e.*, the curve does *not* cut the axis of $x$.

[*These values confirm the drawing of the ellipse if this has been done by aid of the lengths of the axes, but it is better to actually mark on the axes of coordinates the points in which the curve cuts them, and see that they are consistent with the lengths of the axes of the curve before actually drawing the curve.*]

*Example* (ii.). *Trace the curve whose equation is*

$$11x^2+4xy+14y^2-26x-32y+23 = 0.$$

[The following may be regarded as a model solution, but all the devices adopted in Example (i.) should be adopted here.]

(*a*) Here $ab-h^2 = 154-4 = 150.$

Therefore the curve is an ellipse.

(*b*) The equations giving the centre are

$$11x+2y-13 = 0, \quad 2x+14y-16 = 0;$$

whence $x = 1, \quad y = 1.$

(*c*) The equation referred to the centre is

$$11x^2+4xy+14y^2-26\left(\tfrac{1}{2}\right)-32\left(\tfrac{1}{2}\right)+23 = 0$$

or $11x^2+4xy+14y^2 = 6$ or $\frac{11}{6}x^2+\frac{4}{6}xy+\frac{14}{6}y^2 = 1.$

(*d*) The semi-axes are given by

$$\left(A-\frac{1}{r^2}\right)\left(B-\frac{1}{r^2}\right) = H^2$$

or

$$\left(\frac{11}{6}-\frac{1}{r^2}\right)\left(\frac{14}{6}-\frac{1}{r^2}\right) = \left(\tfrac{2}{6}\right)^2 = \tfrac{1}{9}$$

or $\frac{1}{r^4}-\frac{1}{r^2}\cdot\frac{25}{6}+\frac{25}{6} = 0;$

whence $r^2 = \frac{3}{5}$ or $\frac{2}{5}$.

[Equation of curve referred to its principal axes is therefore

$$\frac{x^2}{\frac{3}{5}}+\frac{y^2}{\frac{2}{5}} = 1$$

or $\frac{5x^2}{3}+\frac{5y^2}{2} = 1;$]

$\therefore \quad r = \sqrt{\cdot 6}$ or $\sqrt{\cdot 4}$,

*i.e.* $\cdot 78$ or $\cdot 63$ approximately.

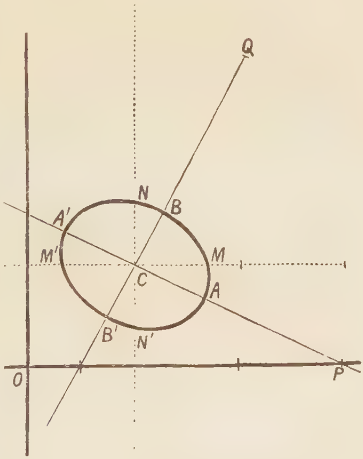



Fig. 43.

(*e*) The equation of the major or minor axis is

$$\left(A-\frac{1}{r^2}\right)x+Hy = 0.$$

When $r^2 = \frac{3}{5}$ (*major* axis), this is

$$\left(\tfrac{11}{6}-\tfrac{5}{3}\right)x+\tfrac{1}{3}y = 0 \quad \text{or} \quad x+2y = 0;$$

when $r^2 = \frac{2}{5}$ (*minor* axis), this is

$$(\tfrac{11}{6} - \tfrac{5}{2})\,x + \tfrac{1}{3}y = 0 \quad \text{or} \quad 2x - y = 0;$$

and the two axes are at right angles, as they should be.

(*f*) Drawing these two lines (Fig. 43) in our diagram, and marking off lengths in both directions equal to the respective semi-axes, we get the points ***A***, ***A′***, ***B***, ***B′*** in the figure.

[It appears the curve does not meet the original axes. This is confirmed by (*g*).]

(*g*) When $x = 0$ or $y = 0$ in the original equation, the corresponding values of $y$ and $x$ are imaginary. Therefore the curve does not meet the original axes. It will, however, meet the lines $x = 1$, $y = 1$, as these pass through the centre.

[Any other convenient lines might be chosen.]

The former meets it in the points

$$\left.\begin{array}{l} \left(1,\ 1 + \dfrac{\sqrt{21}}{7}\right),\quad \left(1,\ 1 - \dfrac{\sqrt{21}}{7}\right) \\ \text{and the latter in } \left(1 + \dfrac{\sqrt{66}}{11},\ 1\right),\quad \left(1 - \dfrac{\sqrt{66}}{11},\ 1\right) \end{array}\right\}$$

or, roughly, $(1,\ 1{\cdot}65)$, $(1,\ {\cdot}35)$, $(1{\cdot}7,\ 1)$, $({\cdot}3,\ 1)$ corresponding to the points ***N***, ***N′***, ***M***, ***M′***; and, putting in these four points, we draw the curve so as to go through them.

*Example* (iii.). *Trace the curve whose equation is*

$$x^2 + xy + y^2 = 12.$$

[Here the origin is the centre, so the process is much shorter.]

$ab - h^2 = 1^2 - (\frac{1}{2})^2 =$ a positive quantity, and therefore curve is an ellipse.

Reducing the absolute term to unity, we write the equation

$$\frac{x^2}{12} + \frac{xy}{12} + \frac{y^2}{12} = 1.$$

The equation giving the semi-axes is

$$\left(\frac{1}{12} - \frac{1}{r^2}\right)\left(\frac{1}{12} - \frac{1}{r^2}\right) = \left(\frac{1}{24}\right)^2 \quad \text{or} \quad \frac{1}{r^4} - \frac{1}{6}\frac{1}{r^2} + \frac{1}{192} = 0$$

whence

$$r^2 = 24 \text{ or } 8;$$

$$\therefore\ r = 2\sqrt{6} \text{ or } 2\sqrt{2} = 4{\cdot}9 \text{ or } 2{\cdot}8, \text{ approx.}$$

The equation of a semi-axis is

$$\left(A - \frac{1}{r^2}\right)x + Hy = 0.$$

$$r^2 = 24 \text{ gives } (\tfrac{1}{12} - \tfrac{1}{24})x + \tfrac{1}{24}y = 0 \quad \text{or} \quad x + y = 0.$$

$$r^2 = 8 \text{ gives } (\tfrac{1}{12} - \tfrac{1}{8})x + \tfrac{1}{24}y = 0 \quad \text{or} \quad x - y = 0,$$

and these lines are at right angles, as they should be.

We now mark the axes in a figure in magnitude and position.
Note also that the lines $x = \pm 4$ and $y = \pm 4$ touch the curve.

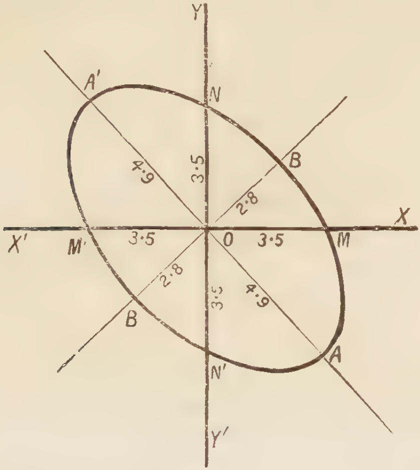



Fig. 44.

The curve cuts $y = 0$ where $x^2 = 12$ or $x = \pm 2\sqrt{3} = \pm 3{\cdot}5$, approx., and similarly it cuts $x = 0$ where $y = \pm 3{\cdot}5$, approx.

These points are indicated by $M$, $M'$, $N$, $N'$ in figure.

Hence the ellipse is as drawn.

## MISCELLANEOUS EXERCISES ON CHAP. VII.

Trace the following curves :—

1. $2x^2 - 2xy + 2y^2 - 2x - 2y - 3 = 0.$
2. $6x^2 - 4xy + 9y^2 - 24x - 22y + 43 = 0.$
3. $5x^2 + 2y^2 + 10x - 12y + 13 = 0.$
4. $17x^2 + 12xy + 22y^2 - 40x - 60y + 24 = 0.$
5. $11x^2 + 6xy + 19y^2 - 28x - 44y - 26 = 0.$
6. $3x^2 + 2\sqrt{2}xy + 2y^2 - 8\sqrt{2}x - 8y + 8 = 0.$
7. $x^2 + xy + y^2 = 6.$

8. Find the equations of the axes of the curves in Exx. 1, 3, 5 referred to the *original* axes.

9. Trace the curve $x^2 + xy + y^2 = 1$, and compare it with the curve obtained in Ex. 7, thus confirming the necessity of the Cautions in § 103.

# CHAPTER VIII.

## TRACING OF HYPERBOLAS.

**115.** We shall now apply the methods of Chap. VI. to the problem of tracing a hyperbola whose equation is given. By reason of the curve being infinite in extent, it is more difficult to trace a hyperbola than an ellipse, although a good deal of the work to be gone through is the same in the two cases. In the hyberbola it is most important, in addition to finding the semi-axes in magnitude and position, to find the asymptotes and trace them; otherwise we cannot be at all certain as to the ultimate direction of the infinite branches.

The method of work is as follows :—

(i.) Find the centre of the curve and the equation when this point is origin.

(ii.) Find the magnitudes of the axes and their equation.

(iii.) Find the asymptotes and trace them.

Then, in addition, it is advisable by way of corroboration to find some points on the curve—those on the original axes of coordinates, if real, will generally suffice, and, if they are not real, we can very easily find lines which do meet the curve in real points.

*Example* (i.). *Trace the curve whose equation is*

$$x^2+10xy+y^2-12x-12y+6=0.$$

[The note appended to Example (i.), p. 106, applies to this example also.]

(*a*) Here $\quad ab-h^2=1-5^2=$ a negative quantity.

Therefore the curve is a hyperbola. (§ 106.)

(*b*) The equations giving the coordinates of the centre are

$$x+5y-6=0, \quad 5x+y-6=0;$$

whence $$x=1, \quad y=1.$$

[*Confirm your work by substituting the values so found.*]

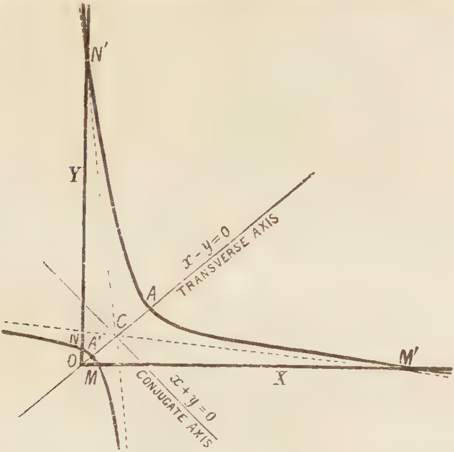



Fig. 45.

(*c*) Substituting half the coordinates of the centre in the terms of the first degree, we get for the equation referred to the centre

$$x^2+10xy+y^2-12\,.\,\tfrac{1}{2}-12\,.\,\tfrac{1}{2}+6=0$$

or $$x^2+10xy+y^2=6$$

or $$\tfrac{1}{6}x^2+\tfrac{10}{6}xy+\tfrac{1}{6}y^2=1. \qquad \text{(See Caution, § 103.)}$$

(*d*) The semi-axes are given by

$$\left(A-\frac{1}{r^2}\right)\left(B-\frac{1}{r^2}\right)=H^2$$

or $$\left(\frac{1}{6}-\frac{1}{r^2}\right)\left(\frac{1}{6}-\frac{1}{r^2}\right)=\left(\frac{5}{6}\right)^2$$

or $$\frac{1}{r^4} - \frac{1}{3} \cdot \frac{1}{r^2} - \frac{2}{3} = 0;$$

whence $$r^2 = 1 \text{ or } -\tfrac{3}{2}.$$

[*In a hyperbola one value of $r^2$ must be negative.*]

This shows that the curve is a hyperbola whose transverse semi-axis is 1, and whose conjugate semi-axis is $\sqrt{\frac{3}{2}}$.

(The equation of the curve referred to its principal axes as axes of coordinates is therefore

$$\frac{x^2}{1} - \frac{y^2}{\frac{3}{2}} = 1 \quad \text{or} \quad x^2 - \tfrac{2}{3}y^2 = 1.)$$

(*e*) The equations of the transverse and conjugate axes are given by

$$\left(A - \frac{1}{r^2}\right)x + Hy = 0.$$

When $r^2 = 1$ (*transverse* axis), this is

$$(\tfrac{1}{6} - 1)\,x + \tfrac{5}{6}y = 0 \quad \text{or} \quad x - y = 0;$$

when $r^2 = -\frac{3}{2}$ (*conjugate* axis), this is

$$(\tfrac{1}{6} + \tfrac{2}{3})\,x + \tfrac{5}{6}y = 0 \quad \text{or} \quad x + y = 0.$$

[*Accuracy of work is confirmed here by the fact that these lines are evidently at right angles.*]

Draw these axes through the centre $C$, and mark off the ends $A$, $A'$ of the transverse axis on the transverse axis $x - y = 0$, $CA$ and $CA'$ being each $= 1$.

[*Do not, in the hyperbola, mark off any lengths corresponding to the value of the conjugate semi-axis, as this would not give any points* ON *the curve. This defect will be fully atoned for by our drawing the asymptotes as in* (*f*).]

(*f*) Next, with the centre as origin, the asymptotes are

$$x^2 + 10xy + y^2 = 0, \qquad (\S\ 110, \text{ Cor. II.})$$

*i.e.*, $$y = (-5 \pm 2\sqrt{6})\,x,$$

or, roughly, $y = -\cdot 1x$ and $y = -9\cdot 9x$ are the two asymptotes.

We can now draw the asymptotes, and we note from the position of the transverse axis that *the curve lies within the obtuse angle* made by them.

[*Confirm your work at this stage by seeing that the axes of the curve appear to bisect the angles between the asymptotes. This is most important.*]

(*g*) Where the line $OX$ meets the curve we have $x = \cdot 5$ or $11\cdot 5$ (giving the points $M$, $M'$), and where $OY$ meets it we have $y = \cdot 5$ or $11\cdot 5$ ($N$, $N'$). Marking these points, there is no difficulty in getting a very fair idea as to the shape of the curve.

[*The reader should be particularly careful to draw the asymptotes in such a case; otherwise he will find it extremely difficult to get the curve right in regions remote from the centre.*]

*Example* (ii.) *Trace the curve whose equation is*

$$3x^2 + 4xy = 6.$$

(*a*) Since $(ab - h^2) = -4$ and is negative, the curve is a hyperbola.

(*b*) The origin is the centre of the curve, since there are no terms of the first degree We therefore proceed at once to find the lengths of the axes

(*c*) Dividing across by 6, the equation is

$$\tfrac{3}{6}x^2 + \tfrac{4}{6}xy = 1.$$

The equation for the semi-axes is

$$\left(A - \frac{1}{r^2}\right)\left(B - \frac{1}{r^2}\right) = H^2$$

or

$$\left(\frac{3}{6} - \frac{1}{r^2}\right)\left(0 - \frac{1}{r^2}\right) = \left(\frac{1}{3}\right)^2$$

or

$$\frac{1}{r^4} - \frac{1}{2}\cdot\frac{1}{r^2} - \frac{1}{9} = 0\,;$$

whence

$$r^2 = -6 \text{ or } \tfrac{3}{2}.$$

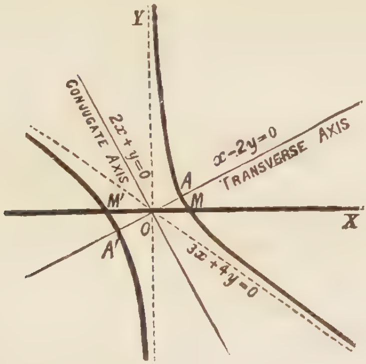



Fig. 46.

Thus the curve is a hyperbola, of which the semi-transverse axis is

$$\sqrt{\tfrac{3}{2}} = \tfrac{1}{2}\sqrt{6} = 1{\cdot}22, \text{ approx.,}$$

and the semi-conjugate axis is $\sqrt{6} = 2{\cdot}45$.

[Equation of curve referred to its principal axes is therefore

$$\frac{x^2}{\frac{3}{2}} - \frac{y^2}{6} = 1 \quad \text{or} \quad \tfrac{2}{3}x^2 - \tfrac{1}{6}y^2 = 1.]$$

(*d*) The equation of an axis being

$$\left(A - \frac{1}{r^2}\right)x + Hy = 0,$$

$r^2 = \frac{3}{2}$ gives $(\frac{1}{2} - \frac{2}{3})x + \frac{1}{3}y$ or $x - 2y = 0$ for the transverse axis;
$r^2 = -6$ gives $(\frac{1}{2} + \frac{1}{6})x + \frac{1}{3}y$ or $2x + y = 0$ for the conjugate axis.
In verification, we note that these are at right angles.
Draw these lines as in figure, and mark off on $x - 2y = 0$,

$$OA = OA' = 1{\cdot}22, \text{ approx.}$$

(*e*) The asymptotes are $3x^2 + 4xy = 0$
or $x = 0$ and $3x + 4y = 0$.

Draw these lines. [Note that the axes of the curve appear to bisect the angles between the asymptotes, as they should.]

(*g*) $y = 0$ meets curve, where

$$3x^2 = 6 \quad \text{or} \quad x = \pm\sqrt{2}, \quad \text{giving points } M, M'.$$

$x = 0$ leads to the peculiar result $0 = 6$; but, on writing equation in the form

$$y = \frac{6 - 3x^2}{4x} = \frac{6}{4x} - \frac{3x}{4},$$

we see that $x = 0$ gives $y = \infty$, *i.e.*, the curve meets the axis of $y$ at an infinite distance. This confirms what we have previously found, viz., that $x = 0$ is an asymptote.

## MISCELLANEOUS EXERCISES ON CHAP. VIII.

Trace the following hyperbolas:—

1. $7x^2 - 60xy + 32y^2 - 106x + 68y - 37 = 0$.
2. $2x^2 + 8xy + 2y^2 - 2x - 2y - 5 = 0$.
3. $6x^2 - 60xy - 19y^2 + 48x + 98y - 60 = 0$.
4. $3x^2 - 8xy - 3y^2 - 4x + 22y - 12 = 0$.
5. $3x^2 - 2y^2 + 6x - 4y - 5 = 0$.
6. $36(x^2 - y^2) + 48x - 36y - 101 = 0$.
7. $7x^2 + 6xy - y^2 = 4$.
8. $6xy + 8y^2 = 3$.
9. $x^2 + 2xy\sqrt{3} - y^2 - x(2 - 4\sqrt{3}) + y(2\sqrt{3} + 4) - 4\sqrt{3} - 1 = 0$.

# CHAPTER IX.

## REDUCTION OF THE GENERAL EQUATION WHEN $ab = h^2$.

116. **General equation when $ab = h^2$.** — We shall now discuss the curve represented by the general equation in the case which we specially excluded in Chap. VI. As we there saw, when $ab = h^2$, we are not able to get rid of the linear terms by taking a new origin, and consequently the method previously used will not apply to this case.

When $ab = h^2$, the terms of the second degree

$$ax^2 + 2hxy + by^2$$

form a perfect square, let us say,

$$(\alpha x + \beta y)^2,$$

so that $$a = \alpha^2, \quad h = \alpha\beta, \quad b = \beta^2,$$

and then the equation is

$$(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0.$$

We have already seen in § 52 that such an equation represents a parabola.

**117. To find (*a*) the axis and tangent at the vertex of the parabola given by the general equation, (*b*) the length of its latus rectum.**

We know that, when a point lies on a fixed parabola, the square of the perpendicular from it on the axis varies directly as the perpendicular on the tangent at the vertex, and, hence, to solve the problem of this article, we must find two lines, bearing such a relation to the curve represented by the equation given, and carefully remember that the two lines have to be at right angles.

When we have found these the length of the latus rectum will enable us to determine the size of the curve.

We shall first of all explain the process by means of a particular example before dealing with the general equation.

*Example.—Find (a) the axis and tangent at the vertex, (b) length of latus rectum of the parabola*

$$16x^2 - 24xy + 9y^2 - 44x - 42y + 49 = 0.$$

(*a*) This equation may be written

$$(4x - 3y)^2 = 44x + 42y - 49 \quad \text{.....................} \quad (\text{A}).$$

Now $4x - 3y$ and $44x + 42y - 49$ are proportional to the perpendiculars from $(x, y)$ on to the lines $4x - 3y = 0$ and $44x + 42y - 49 = 0$.

Equation (A) therefore states that the square of the perpendicular from any point of the curve on $4x - 3y = 0$ varies as the perpendicular from that point on $44x + 42y - 49 = 0$. If these two lines were at right angles, they would be the two required (§ 41). They are, however, not perpendicular, but we can attain our result by the following manipulation of (A).

Introduce the quantity $\lambda$ into the left-hand side of (A), thus

$$(4x - 3y + \lambda)^2 = \dots .$$

This results in the addition of $8\lambda x - 6\lambda y + \lambda^2$ to the left-hand side; so that for (A) still to be true we must add the same quantity to the right-hand side. Thus we get

$$(4x - 3y + \lambda)^2 = x(44 + 8\lambda) + y(42 - 6\lambda) + \lambda^2 - 49.$$

*Now choose* $\lambda$ *so that the lines*

$$4x - 3y + \lambda = 0, \quad x(44 + 8\lambda) + y(42 - 6\lambda) + \lambda^2 - 49 = 0$$

*are at right angles.*

We must then have

$$4(44 + 8\lambda) - 3(42 - 6\lambda) = 0 \text{ or } 50 + 50\lambda = 0; \quad (\text{Part I. §19})$$

$$\therefore \quad \lambda = -1.$$

Hence the equation may be written

$$(4x - 3y - 1)^2 = 36x + 48y - 48 = 12(3x + 4y - 4) \quad \text{......} \quad (\text{B}).$$

The lines $4x - 3y - 1 = 0$ and $3x + 4y - 4 = 0$ are at right angles. Hence (B) affirms that the *square of the perpendicular* from a point of the curve on to the line $4x - 3y - 1 = 0$ varies as the perpendicular from the same point on to $3x + 4y - 4 = 0$, which is perpendicular to the former.

Hence (§ 41) $4x - 3y - 1 = 0$ is the axis of the curve, and $3x + 4y - 4 = 0$ is the tangent at the vertex.

**Caution.**—Students are frequently at a loss to decide which of the two equations represents the axis, and which the tangent at the vertex. This difficulty can be diminished by a comparison with $Y^2 = 4aX$, where $Y = 0$ (the axis of $X$) is the axis of the curve. Thus the "square" quantity corresponds to the axis of the curve.

($b$) The square of the perpendicular on $4x-3y-1 = 0$ is

$$\frac{(4x-3y-1)^2}{25},$$

and the perpendicular on $3x+4y-4 = 0$ is

$$\frac{3x+4y-4}{5}.$$

Therefore the latus rectum $2l$ or $4a$ is given by

$$\frac{(4x-3y-1)^2}{25} = 2l\frac{3x+4y-4}{5} \quad \text{.................} \quad (\text{C});$$

but, since $(x,y)$ is in the curve,

$$(4x-3y-1)^2 = 12(3x+4y-4) \quad \text{..................} \quad (\text{B})$$

From (C) and (B) we get by division

$$\therefore \quad \tfrac{12}{25} = 2l \cdot \tfrac{1}{5};$$

$$\therefore \quad \text{latus rectum} = \tfrac{12}{5}.$$

Since $(4x-3y-1)^2$ is positive, it follows that, for every point of the curve, the quantity $3x+4y-4$ must be positive, so that the curve lies on that side of the tangent at the vertex $3x+4y-4=0$ for which $3x+4y-4$ is positive.

But the origin is on that side of $3x+4y-4=0$ for which $3x+4y-4$ is negative (Part I., § 13).

Hence the origin and the curve are on *opposite* sides of $3x+4y-4=0$.

This last point will be found to be of great importance when we come to tracing a parabola.

**Caution.**—If equation (B) had appeared in the form

$$(4x-3y-1)^2 = -12(3x+4y-4),$$

for every point of the curve $-12(3x+4y-4)$ would have had to be positive, *i.e.*, $3x+4y-4$ would have had to be negative, and then the curve and origin would have been on the *same* side of $3x+4y-4=0$.

We shall see further illustrations of this point in the next chapter.

The reader should now work Exx. 1–7 at the end of the chapter.

**118. General case.** The general equation, being written

$$(\alpha x+\beta y)^2 = -2gx-2fy-c,$$

shows at once that the square of the perpendicular from a

point on the curve on the line $\alpha x+\beta y=0$ varies as the perpendicular on the line $2gx+2fy+c=0$ (§§ 51, 52).

If these two lines were at right angles, they would be the two required; they are, however, generally *not* at right angles, so we put the equation in the form

$$(\alpha x+\beta y+\lambda)^2=(-2gx-2fy-c)+(2\alpha\lambda x+2\beta\lambda y+\lambda^2)$$
$$=-2x(g-\alpha\lambda)-2y(f-\beta\lambda)-c+\lambda^2.$$

We now see that, whatever $\lambda$ may be, the square of the perpendicular on $\alpha x+\beta y+\lambda=0$ varies as the perpendicular on

$$2x(g-\alpha\lambda)+2y(f-\beta\lambda)+c-\lambda^2=0,$$

and *we therefore proceed to choose $\lambda$ so that these two lines are at right angles.* The condition for this is

$$\alpha(g-\alpha\lambda)+\beta(f-\beta\lambda)=0 \quad \text{or} \quad \lambda=\frac{\alpha g+\beta f}{\alpha^2+\beta^2}.$$

When $\lambda$ has this value, the square of the perpendicular on $\alpha x+\beta y+\lambda=0$ varies as the perpendicular on

$$2x(g-\alpha\lambda)+2y(f-\beta\lambda)+c-\lambda^2=0,$$

and the two lines are at right angles; hence the equation represents a parabola having the former line for axis and the latter for tangent at the vertex.

**Caution.**—In applying this process to an example, the reader must follow out the general reasoning, and not quote the formulæ.

119. **To find the latus rectum of the above curve.**

If $P$ be a point on the curve, $PM$ the perpendicular on the tangent at the vertex, and $PN$ that on the axis, we have

$$PN^2=2l\,.\,PM,$$

where $2l$ is the latus rectum.

Now, using the equations above, we have

$$PN=\frac{\alpha x+\beta y+\lambda}{\sqrt{\alpha^2+\beta^2}},$$

$$PM=\frac{2x(g-\alpha\lambda)+2y(f-\beta\lambda)+c-\lambda^2}{2\sqrt{(g-\alpha\lambda)^2+(f-\beta\lambda)^2}},$$

and
$$\lambda=\frac{\alpha g+\beta f}{\alpha^2+\beta^2};$$

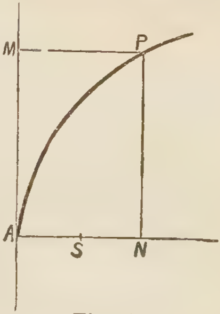



Fig. 47.

$$\therefore \quad \frac{(\alpha x+\beta y+\lambda)^2}{\alpha^2+\beta^2} = 2l\,\frac{2x(g-\alpha\lambda)+2y(f-\beta\lambda)+c-\lambda^2}{2\sqrt{(g-\alpha\lambda)^2+(f-\beta\lambda)^2}}.*$$

But $(x, y)$ is on the curve; therefore

$$(\alpha x+\beta y+\lambda)^2 = -2x(g-\alpha\lambda)-2y(f-\beta\lambda)-c+\lambda^2;$$

hence by division

$$2l = -\frac{2\sqrt{(g-\alpha\lambda)^2+(f-\beta\lambda)^2}}{\alpha^2+\beta^2},$$

where
$$\lambda = \frac{\alpha g+\beta f}{\alpha^2+\beta^2}.$$

(The sign is of no importance, since we want merely the *length* of the latus rectum.)

Now $(g-\alpha\lambda)^2+(f-\beta\lambda)^2$

$$= g^2+f^2-\lambda(\alpha g+\beta f)+\lambda\left\{\lambda(\alpha^2+\beta^2)-(\alpha g+\beta f)\right\}$$

$$= \frac{(g^2+f^2)(\alpha^2+\beta^2)-(\alpha g+\beta f)^2}{\alpha^2+\beta^2}$$

$$[\text{for } \lambda(\alpha^2+\beta^2)-(\alpha g+\beta f)=0]$$

$$= \frac{(\alpha f-\beta g)^2}{\alpha^2+\beta^2}.$$

$$\therefore \quad 2l = \frac{2(\alpha f-\beta g)}{(\alpha^2+\beta^2)^{\frac{1}{2}}(\alpha^2+\beta^2)}$$

$$= \frac{2(\alpha f-\beta g)}{(\alpha^2+\beta^2)^{\frac{3}{2}}}.$$

But
$$\alpha = \sqrt{a}, \quad \beta = \sqrt{b};$$

$$\therefore \quad 2l = \frac{2(f\sqrt{a}-g\sqrt{b})}{(a+b)^{\frac{3}{2}}}.$$

We have given the above with more than necessary fulness in order to recapitulate the process to some extent; it is evident, *a priori*, that the numerators in the functions first written down are equal, and hence we can cancel without writing them down.

---

* Comparing with $Y^2=4aX$, in which $Y=0$ is the axis, we see that $\alpha x+\beta y+\lambda=0$ is the axis here. (§ 41.)

**120. Exceptional case.**

In discussing the general equation of the parabola, we found that, for a point on the curve,

$$(\alpha x + \beta y + \lambda)^2 = -2x(g - \alpha\lambda) - 2y(f - \beta\lambda) - c + \lambda^2,$$

and the two lines

$$\alpha x + \beta y + \lambda = 0 \quad \text{and} \quad 2x(g - \alpha\lambda) + 2y(f - \beta\lambda) + c - \lambda^2$$

are at right angles when $\qquad \lambda = \dfrac{\alpha g + \beta f}{\alpha^2 + \beta^2}.$

If, now, $\dfrac{g}{\alpha} = \dfrac{f}{\beta}$, each $= \dfrac{\alpha g + \beta f}{\alpha^2 + \beta^2} = \lambda$ ;

$$\therefore \quad g - \alpha\lambda = 0 \quad \text{and} \quad f - \beta\lambda = 0.$$

Therefore the right-hand side becomes a constant, and the rest of the proof breaks down.

We then have $\qquad (\alpha x + \beta y + \lambda)^2 = \lambda^2 - c$ ;

or $\qquad \alpha x + \beta y + \lambda = \pm\sqrt{\lambda^2 - c},$

which obviously represents two parallel straight lines.

Hence the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel straight lines if

$$ab = h^2 \quad \text{and} \quad g\beta = f\alpha,$$

*i.e.*, $\qquad$ if $ab = h^2$ and $g\sqrt{b} = f\sqrt{a}$,

*i.e.*, $\qquad$ if $ab = h^2$ and $af^2 = bg^2$.

This case is very easily recognized at sight; for, since $f/g = \alpha/\beta$, it follows that $\alpha x + \beta y$ differs from $gx + fy$ only by a numerical factor, and the equation is of the form

$$(\alpha x + \beta y)^2 + P(\alpha x + \beta y) + C = 0.$$

Solving the quadratic for $\alpha x + \beta \gamma$, we get two values, and hence two parallel lines.

Cor. *If, in addition, $\lambda^2 = c$, the two lines coincide.*

*Example.—Find what the equation*

$$9x^2 + 24xy + 16y^2 - 9x - 12y + 2 = 0$$

*represents.*

This may be written

$$(3x + 4y)^2 - 9x - 12y + 2 = 0,$$

and therefore falls under the head of this chapter.

Since $$9x + 12y = 3(3x + 4y),$$
it may be simplified further, thus:
$$(3x + 4y)^2 - 3(3x + 4y) + 2 = 0 \quad \text{or} \quad (3x + 4y - 2)(3x + 4y - 1) = 0,$$
and thus represents two parallel lines
$$3x + 4y - 2 = 0 \quad \text{and} \quad 3x + 4y - 1 = 0.$$

If we had proceeded just as in the parabola, we should ultimately have arrived at this result; but by noticing the relation at first we shorten the work.

## MISCELLANEOUS EXERCISES ON CHAP. IX.

Find the axis and tangent at the vertex of each of the following parabolas*:—

1. $y^2 = 3x + 2$.
2. $y^2 = x + y + 1$.
3. $x^2 = x + y + 1$.
4. $(x + y)^2 = 3(x - y + 1)$.
5. $(x + y)^2 - x + 5y + 4 = 0$.
6. $9x^2 + 24xy + 16y^2 - 98x + 11y - 94 = 0$.

7. Find the latera recta of the parabolas 1–6.

Discuss the curves represented by the following equations:—

8. $x^2 + 2xy + y^2 = 4$.
9. $x^2 - 2xy + y^2 + 7x - 7y + 6 = 0$.
10. $(px + qy)^2 + rx + sy + t = 0$, where $\dfrac{r}{s} = \dfrac{p}{q}$.
11. $4x^2 + 4xy + y^2 + 4x + 2y + 1 = 0$.

12. Show that the equation $(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$ represents two parallel lines if $\alpha f - \beta g = 0$.

Verify that the condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ is satisfied in this case.

* The results of Exx. 1–7 will be required in working the Exercises of Chap. X

# CHAPTER X.

## TRACING OF PARABOLAS.

**121.** We shall now give some applications of the methods of last chapter to tracing parabolas whose equations are given. If the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a parabola, then $ab=h^2$.

Having satisfied ourselves that this is the case, we use the method of last chapter to find the tangent at the vertex, the axis, and the latus rectum. The tracing is not then a difficult matter, but, by way of verification, we always find a few points on the curve—*e.g.*, those on the axes of coordinates; if these are imaginary, we can easily choose some line which meets the curve in real points.

*Example* (i.) *Trace the curve whose equation is*

$$x^2+2xy+y^2-6x-2y+4=0.$$

(*a*) Here $\qquad ab-h^2=1.1-1^2=0;$

so the curve is a parabola.

(*b*) The square of the perpendicular on the line $x+y=0$ varies as that on the line $6x+2y-4=0$, since

$$(x+y)^2=6x+2y-4.$$

These two lines are not at right angles, but the equation may be written $\qquad (x+y+\lambda)^2=x(6+2\lambda)+y(2+2\lambda)-4+\lambda^2,$

and the lines $x+y+\lambda=0$, $x(6+2\lambda)+y(2+2\lambda)-4+\lambda^2=0$

are at right angles if $\qquad 6+2\lambda+2+2\lambda=0,$

*i.e.*, if $\qquad \lambda=-2.$

(*c*) Hence we have $\qquad (x+y-2)^2=2(x-y)$ ........................ (A),

and the lines $x+y-2=0$, $x-y=0$ are at right angles.

Comparing with $Y^2=4aX$, we see that $x+y-2=0$ is the axis, and $x-y=0$ the tangent at the vertex.

(*d*) The latus rectum $4a$ is given by

$$(\text{perp. on } x+y-2=0)^2 = 4a\,(\text{perp. on } x-y=0)$$

or
$$\left(\frac{x+y-2}{\sqrt{1^2+1^2}}\right)^2 = 4a\ \frac{x-y}{\sqrt{1^2+1^2}};$$

$$\therefore\ 4a=\sqrt{2},\quad \text{since}\quad (x+y-2)^2 = 2\,(x-y).$$

(*e*) The left-hand side of equation (A), being a square, is positive; hence the curve lies entirely on that side of the tangent $x-y=0$ for which $x-y$ is positive, *i.e.*, $x>y$. This is clearly the *lower* side.

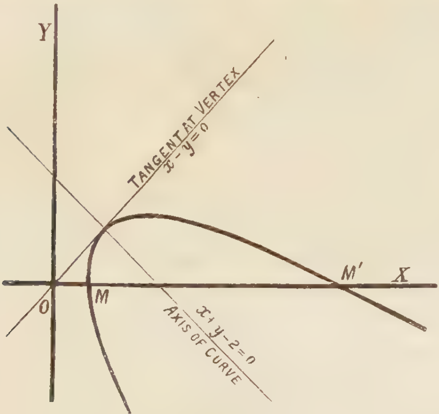



Fig. 48.

(*f*) The curve meets $x=0$, where

$$y^2-2y+4=0;$$

whence $y$ is imaginary.

Therefore curve does not meet axis of $y$.

It meets $y=0$, where

$$x^2-6x+4=0 \quad\text{or}\quad x=3\pm\sqrt{5}=5{\cdot}24 \text{ or } {\cdot}76,\quad \text{approx.},$$

corresponding to points ***M***, ***M′*** in diagram.

**Caution.**—Theoretically, when the length of the latus rectum and the equations of the axis and tangent at vertex have been found, we have sufficient information to enable us to draw the curve; but, in actual practice, it is much easier to disregard the length of the latus rectum, and find some other points on the curve, as, for example, in (*f*) above.

The fact that the curve is symmetrical with respect to its axis is of great assistance, but, to get a good idea of the size of the curve, a number of points on it should always be marked before tracing.

*Example* (ii.). *Trace the curve* $x^2-2xy+y^2-2x-2y+4=0$.

(*a*) Here

$ab-h^2 = 1 \cdot 1-(-1)^2 = 0$;

so the curve is a parabola.

(*b*) The equation may be written

$$(x-y)^2 = 2(x+y-2),$$

showing that the square of the perpendicular on the line $x-y=0$ varies as the perpendicular on the line $x+y-2=0$.

These two lines are actually at right angles, so we at once infer that

the axis of the curve is $x-y=0$,

and the tangent at the vertex is $\quad x+y-2=0$.

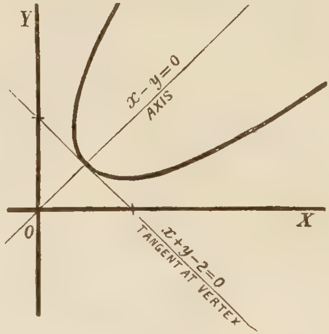



Fig. 49.

(*c*) Further, the curve lies on that side of the latter line for which $x+y-2$ is positive, while the origin lies on the side for which $x+y-2$ is negative, *i.e.*, the origin and curve are on opposite sides of $x+y-2=0$.

(*d*) To find the latus rectum $4a$, we have

$$(\text{perp. on } x-y=0)^2 = 4a\,(\text{perp. on } x+y-2=0),$$

or
$$\left(\frac{x-y}{\sqrt{2}}\right)^2 = 4a\,\frac{x+y-2}{\sqrt{2}},$$

but by the equation of the curve $(x-y)^2 = 2(x+y-2)$. Hence $4a=\sqrt{2}$, and this is the length of the latus rectum.

(*e*) The axes will be found to meet the curve in imaginary points. We should therefore find the points of the curve in which it is met by other straight lines. Since the vertex is at (1, 1) (the point of intersection of $x-y=0$ and $x+y-2=0$), it is clear that, when $x>1$, $y$ is real, so we can find any number of points. Thus $x=1$ gives $y=1$ or 3, $x=2$ gives $y=3\pm\sqrt{5}$, and before drawing the curve we should mark these points on the figure.

## MISCELLANEOUS EXERCISES ON CHAP. X.

1. Trace all the parabolas in the exercises in Chap. IX.

Trace the following parabolas, finding the axis, tangent at the vertex, and latus rectum of each:—

2. $4x^2-4xy+y^2+2x-26y+9=0$.
3. $x^2+6xy+9y^2+15x-5y+125=0$.
4. $4x^2+12xy+9y^2-2x+10y+21=0$.
5. $25x^2-40xy+16y^2+52x-17y+16=0$.
6. $(7x+9y)^2+23x+11y+12=0$.

# CHAPTER XI.

## TRACING OF CONICS FROM THEIR EQUATIONS.

122. In this chapter we shall solve some miscellaneous examples of tracing curves of the second degree by availing ourselves of the principles explained in Chapters VI. and IX.

It is, of course, always possible to trace a curve whose equation is given, for we can obtain as many points as we like on it, but this expedient alone would be very tedious; for example, in the case of the hyperbola, for we should need a very large number of points to be sure of getting the curve correctly. Accordingly, in practice we apply the general principles set forth in the preceding chapters to acquire a *general* idea of the shape and position of the curve.

123. We shall begin with some general directions which will serve also as a convenient summary of the results of Chapters VI. and IX. To some extent we recapitulate what has been said in Chapters VII, VIII., X.

Suppose the equation to be, as usual,

$$ax^2+2hxy+by^2+2gx+2fy+c=0;$$

then the following method may be used in tracing the curve:—

(1) From the value of $ab-h^2$, find the nature of the curve, viz., $ab-h^2$ is positive for an ellipse, zero for a parabola, and negative for a hyperbola.

(2) If $ab=h^2$, then the tangent at the vertex, the axis, and the latus rectum must be found as in Chapter IX.,

and care must be taken to place the curve on the right side of the tangent at the vertex.

(3) If $ab > h^2$, then find the centre of the ellipse, and the directions and lengths of the semi-axes. The tracing is then easy from the simple shape of the curve, but care must be taken to associate with a length of a semi-axis the right direction as determined by § 103.

(4) If $ab < h^2$, it is quite essential to find the asymptotes and draw them after finding the centre of the curve. The equation referred to the centre as origin must also be obtained, and the directions and lengths of the semi-axes as in the case of the ellipse. (The fact that the axes bisect the angles between the asymptotes affords a good test of the accuracy of numerical work, in which the beginner is very apt to make slips.)

If the expression $ax^2 + 2hxy + by^2$ has rational factors (which, in general, is not the case), the equations of the asymptotes may be found separately by solving the equation for the two as a quadratic in $y$, or guessing its factors in any way. To get the asymptotes, we have thus to draw lines parallel to $ax^2 + 2hxy + by^2 = 0$ through the centre which we have found already. If the factors are not rational, we may trace the asymptotes from their joint equation very easily, for we have only to find the points in which they meet one of the axes, and then join these points to the centre.

As we have already given examples of drawing the asymptotes in the first way, in this chapter we shall give examples of the second. The first method is, however, the one the reader should use in beginning the subject.

If the asymptotes are $POP'$ and $QOQ'$, the curve may lie either in the angular spaces $POQ$ and $P'OQ'$ or in the angular spaces $POQ'$ and $P'OQ$. As we know which axis is the transverse axis, we can at once settle this, or finding a single point on the curve will suffice also. After settling this, we should mark a number of points on the curve, as, for example, those in which it meets the axes. For others give $x$ the values 1, 2, 3, ..., and solve for $y$.

It will be noticed that we do not use the ordinary test for a pair of straight lines. There is no need to do this,

for we shall see that the equation does represent lines when we refer it to its centre as origin, for the constant term will then be zero. Thus, in using the test $ab-h^2$, a pair of real lines is included under the head of hyperbola, a pair of imaginary lines under ellipse, and a pair of parallel lines under parabola.

We shall now give some examples.

*Example* (i.). *Trace the conic whose equation is*

$$6x^2+4xy+6y^2-20x+4y+14=0.$$

Here $a=6$, $b=6$, $h=2$ so that $ab-h^2$ is positive and the curve is an ellipse.

The equations giving the centre are

$$6x+2y-10=0 \quad \text{and} \quad 2x+6y+2=0,$$

leading to $$x=2, \quad y=-1.$$

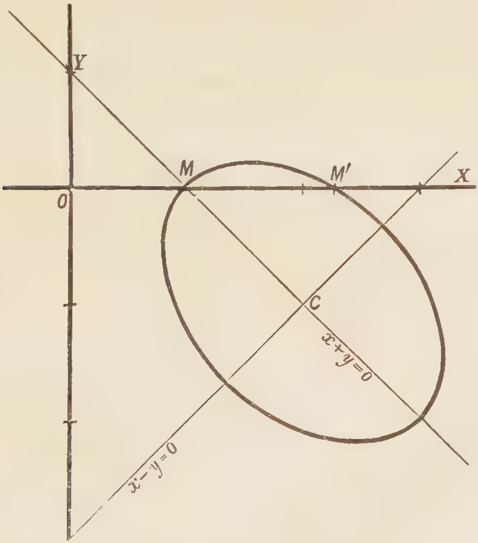



Fig. 50.

To get the equation referred to the centre, we substitute half the coordinates of the centre in the terms of the first degree. The equation is, therefore,

$$6x^2+4xy+6y^2-20\times 1+4\left(-\tfrac{1}{2}\right)+14=0$$

or $$6x^2+4xy+6y^2=8 \quad\text{or}\quad \tfrac{3}{4}x^2+\tfrac{2}{4}xy+\tfrac{3}{4}y^2=1.$$

Hence the semi-axes are given by

$$\left(A-\frac{1}{r^2}\right)\left(B-\frac{1}{r^2}\right)=H^2$$

or $$\left(\frac{3}{4}-\frac{1}{r^2}\right)\left(\frac{3}{4}-\frac{1}{r^2}\right)=\left(\frac{1}{4}\right)^2 \quad\text{or}\quad \frac{3}{4}-\frac{1}{r^2}=\pm\frac{1}{4},$$

from which $$\frac{1}{r^2}=1 \text{ or } \tfrac{1}{2};$$

$$\therefore\quad r=1 \text{ or } \sqrt{2}.$$

The equation of the first axis (referred to the centre as origin, of course) is $\left(\tfrac{3}{4}-1\right)x+\tfrac{1}{4}y=0$ or $x-y=0$,
and the other is $\left(\tfrac{3}{4}-\tfrac{1}{2}\right)x+\tfrac{1}{4}y=0$ or $x+y=0$.

For the points of intersection with $OX$ we have

$$6x^2-20x+14=0 \quad\text{or}\quad x=1 \text{ or } \tfrac{7}{3} \quad (M \text{ and } M' \text{ in figure}),$$

and it is easy to see that $OY$ meets the curve in imaginary points.

Hence, marking the centre and drawing the semi-axes, and then noting the points of intersection with $OX$, the curve can be readily drawn.

The reader should note that one end of the major axis is on $OX$.

*Example* (ii.). *Trace the curve whose equation is*

$$9x^2-24xy+16y^2+32x-76y+16=0.$$

Since here $$ab-h^2=9.16-(12)^2=0,$$
the curve is a parabola. Writing the equation in the form

$$(3x-4y)^2=-32x+76y-16,$$

and introducing $\lambda$ in the usual manner, we have

$$(3x-4y+\lambda)^2=-2x(16-3\lambda)+2y(38-4\lambda)-16+\lambda^2.$$

The two lines

$$3x-4y+\lambda=0 \quad\text{and}\quad -2x(16-3\lambda)+2y(38-4\lambda)-16+\lambda^2=0$$

are at right angles if

$$3(16-3\lambda)+4(38-4\lambda)=0, \quad \textit{i.e.} \text{ if } \lambda=8.$$

Hence the axis is $$3x-4y+8=0,$$
and the tangent at the vertex

$$4(4x+3y+12)=0 \quad\text{or}\quad 4x+3y+12=0,$$

the final form of the equation being

$$(3x-4y+8)^2=4(4x-3y+12).$$

Thus the curve is wholly on the positive (*i.e.*, the origin) side of

$$4x-3y+12=0,$$

the tangent at the vertex, as also follows from the points where it meets the axis. The latus rectum is given by

$$\left(\frac{3x-4y+8}{5}\right)^2 = 2l\,\frac{4x-3y+12}{5}$$

but

$$(3x-4y+8)^2 = 4\,(4x-3y+12)$$

$$\therefore\quad 2l = \tfrac{4}{5}.$$

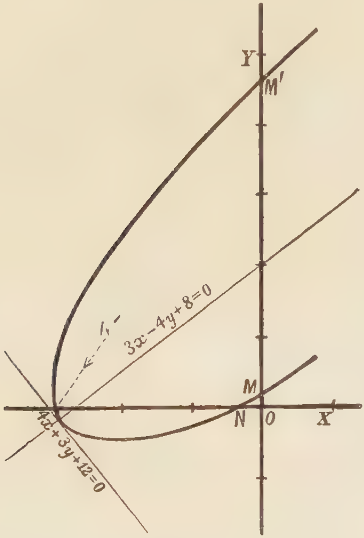



Fig. 51.

Where it cuts the axis of $x$ we have

$$9x^2+32x+16 = 0;$$

$$\therefore\quad x = \frac{-16\pm\sqrt{16^2-16.9}}{9}$$

$$= -\cdot 6 \text{ or } -2\cdot 9, \text{ roughly } (N,\ N' \text{ in Fig. 51}).$$

Where it cuts $OY$, $\quad 4y^2-19y+4 = 0;$

$$\therefore\quad y = \frac{19\pm\sqrt{297}}{8}$$

$$= \cdot 2 \text{ or } 4\cdot 5\ (M,\ M' \text{ in Fig. 51}).$$

Constructing the tangent and axis, then, knowing the points in which it meets the axis, we can draw the curve fairly accurately.

*Example* (iii.). *Trace the curve whose equation is*

$$2x^2-2y^2-xy-6x-7y-4 = 0.$$

Here $ab-h^2$ is negative and the curve is a hyperbola.

Since here $a = 2$, $h = -\frac{1}{2}$, $b = -2$, $g = -3$, $f = -\frac{7}{2}$, $c = -4$, the equations giving the centre are

$$2x-\tfrac{1}{2}y-3 = 0, \qquad -\tfrac{1}{2}x-2y-\tfrac{7}{2} = 0,$$

leading to $$x = 1, \quad y = -2.$$

Referred to the centre $C$ as origin, the equation is

$$2x^2-2y^2-xy-6\left(\tfrac{1}{2}\right)-7(-1)-4 = 0 \quad \text{or} \quad 2x^2-2y^2-xy = 0,$$

and hence the equation represents two straight lines at right angles.

[See Caution below.]

Where the curve meets $OX$ we have

$$2x^2-6x-4 = 0;$$

$$\therefore \quad x = \frac{3 \pm \sqrt{17}}{2}$$

$$= 3{\cdot}5 \text{ or } -{\cdot}5,$$

roughly ($M$ and $M'$);

and where it meets $OY$

$$2y^2+7y+4 = 0;$$

$$\therefore \quad y = \frac{-7 \pm \sqrt{17}}{4}$$

$$= -2{\cdot}8 \text{ or } -{\cdot}7,$$

roughly ($N$ and $N'$).

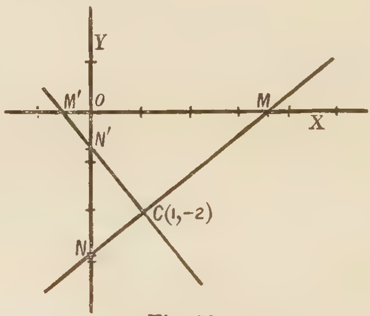



Fig. 52.

Thus the equation represents two lines meeting in the point $(1, -2)$, so that, joining this point to the points of intersection with either axis, we have the curve required.

**Caution.**—If, in the course of our work, the curve appears to reduce to two straight lines, we at once apply to the original equation the test for two straight lines, viz.,

$$abc+2fgh-af^2-bg^2-ch^2 = 0.$$

Here $abc+2fgh-af^2-bg^2-ch^2$

$= 2(-2)(-4)+2\left(-\frac{7}{2}\right)(-3)\left(-\frac{1}{2}\right)-2\left(-\frac{7}{2}\right)^2-(-2)(-3)^2-(-4)\left(-\frac{1}{2}\right)^2$

$= 16-10\frac{1}{2}-24\frac{1}{2}+18+1 = 0.$

Hence the accuracy of our work is confirmed.

*Example* (iv.). *Trace the conic*

$$(x+2y-2)^2+4(2x-y+1)^2=45.$$

We notice that the two lines

$$x+2y-2=0 \quad \text{and} \quad 2x-y+1=0$$

are at right angles, and that the expressions in brackets are *proportional* to the perpendiculars from a point $P(x, y)$ on the curve to these lines.

Making the expression in the brackets actually equal to such perpendiculars, we get

$$\left(\frac{x+2y-2}{\sqrt{5}}\right)^2+4\left(\frac{2x-y+1}{\sqrt{5}}\right)^2$$
$$=\tfrac{45}{5}=9.$$

If, therefore, we take

$$\left.\begin{array}{r} x+2y-2=0 \\ \text{and} \qquad 2x-y+1=0 \end{array}\right\} \quad \ldots\ldots \text{(A)}$$

as axes, calling them the axis of $X$ and $Y$ respectively, the equation reads

$$\frac{(\text{perp. from } P \text{ on axis of } X)^2}{9}$$
$$+\frac{(\text{perp. from } P \text{ on axis of } Y)^2}{\frac{9}{4}}=1.$$

This evidently is an ellipse, with the lines (A) as axes, and the length of its semi-axes are 3 and $\frac{3}{2}$.

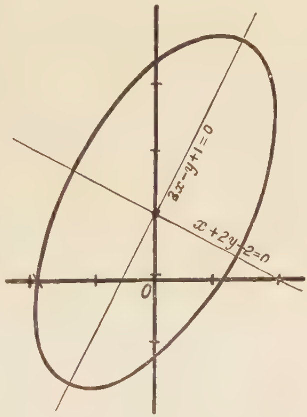



Fig. 53.

**Caution.**—By a comparison with $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, it is seen that 3 is the length of the semi-axis measured along the line corresponding to $y=0$, *i.e.* along $2x-y+1=0$. This point should be noticed very carefully.

To confirm the work, we find the intercepts on the original axes.

$x=0$ gives $(2y-2)^2+4(y-1)^2=45$, whence $y=3{\cdot}36$ or $-1{\cdot}36$;

$y=0$ gives $(x-2)^2+4(2x+1)^2=45$, whence $x=-1{\cdot}9$ or $1{\cdot}2$.

**Caution.**—The above method is of use only when the equation is given in the form $\qquad aR^2+bS^2=\text{constant},$

where $R=0$ and $S=0$ are two straight lines at right angles. The process is analogous to the method of procedure in the case of the parabola.

*Example* (v.). *Trace the curve* $xy+3x+4y+4=0$.

Here $ab-h^2 = 0-(\frac{1}{2})^2$ and is negative.

Therefore the equation represents a hyperbola.

The ordinary processes can be shortened here, for we can write the equation in the form $(x+4)(y+3)=8$,
and now we see that it represents a rectangular hyperbola having $x+4=0$ and $y+3=0$ for asymptotes, and $-4$, $-3$ for centre. (In fact the product of the perpendiculars on these lines is constant.)

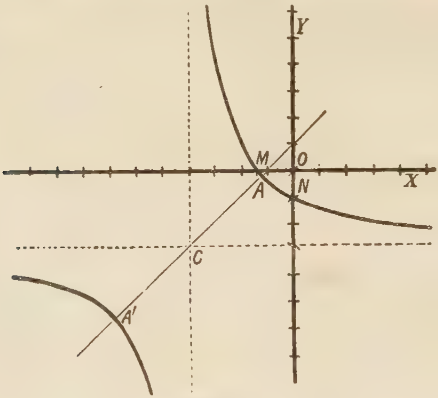



Fig. 54.

Referred to the centre as origin, the equation is

$$xy = 8$$

[for, in order to transfer to $(-4, -3)$, we must write $x-4$ for $x$ and $y-3$ for $y$]. From this the tracing is easy, for, giving $x$ successive values 1, 2, 3, 4, ..., $-1$, $-2$, $-3$, ..., we can write down the values of $y$ and trace the curve.

The equation giving the semi-axes is

$$\frac{1}{r^4} - \left(\frac{1}{16}\right)^2 = 0,$$

for $A = B = 0$ and $H = \frac{1}{16}$.

$$\therefore \quad r^2 = \pm 16.$$

Thus the real semi-axis is of length 4, and its direction referred to the centre as origin is $x-y=0$. This gives the points $A, A'$. The other axis is of course the line $x+y=0$.

The curve meets the original axes in

$y=0$, $x=-\frac{4}{3}$ ($M$, Fig. 54), and $x=0$, $y=-1$ ($N$, Fig. 54).

Hence the curve is clearly as in Fig. 54.

*Example* (vi.). *Trace the curve whose equation is*

$$x^2+4xy+4y^2+7x+14y+6=0.$$

This clearly satisfies the condition for a parabola.

Writing it in the form

$$(x+2y)^2+(7x+14y+6)=0,$$

we introduce $\lambda$ in the usual manner, viz.:

$$(x+2y+\lambda)^2=-x(7-2\lambda)-y(14-4\lambda)-6+\lambda^2,$$

and the two lines $\qquad x+2y+\lambda=0,$

$$x(7-2\lambda)+y(14-4\lambda)+6-\lambda^2=0$$

are at right angles if

$$7-2\lambda+28-8\lambda=0, \quad i.e. \text{ if } \quad \lambda=3\tfrac{1}{2}.$$

Hence the equation becomes

$$(x+2y+3\tfrac{1}{2})^2=12\tfrac{1}{4}-6=6\tfrac{1}{4},$$

so that it represents two parallel straight lines, viz.:

$$x+2y+3\tfrac{1}{2}=\pm 2\tfrac{1}{2},$$

or $\qquad x+2y+1=0 \quad \text{and} \quad x+2y+6=0$

[*Confirm this result by applying the* $abc+2fgh-\ldots$ *test.*]

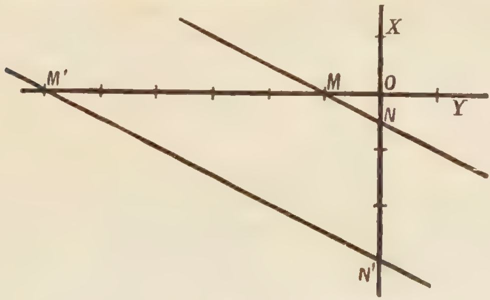



Fig. 55.

The curve cuts original axis $OX$ where $x=-1$ or $-6$ ($M$, $M'$), and $OY$ where $y=-\frac{1}{2}$ or $-3$ ($N$, $N'$).

Knowing these intersections, it is an easy matter to draw the straight lines.

That the equation represents two parallel straight lines is evident, *a priori*, for we may write it in the form

$$(x+2y)^2+7(x+2y)+6=0,$$

or $\qquad (x+2y+1)(x+2y+6)=0.$

The above analysis shows that the true nature of the curve is discovered easily enough in the ordinary process.

*Example* (vii.). *Trace the curve whose equation is*

$$4x^2+12xy-y^2-40x-20y+24=0.$$

Since $ab-h^2=4(-1)-36$ and is negative, this curve is a hyperbola.

The equations giving the centre are

$$4x+6y-20=0,$$
$$6x-y-10=0,$$

leading to $x=2,\quad y=2.$

The equation of the curve with the centre as origin is

$$4x^2+12xy-y^2=36$$

or $\frac{4}{36}x^2+2\,.\,\frac{1}{6}xy-\frac{1}{36}y^2=1$;

so that the equation for the semi-axes is

$$\frac{1}{r^4}-\frac{1}{r^2}\cdot\frac{1}{12}-\frac{5}{162}=0,$$

leading to $r^2=\frac{36}{8}$ or $-\frac{36}{5}$.

Thus the length of the semi-transverse axis is

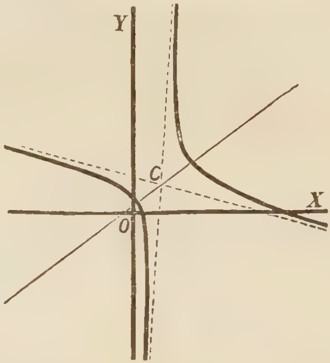



Fig. 56.

$$\frac{6}{2\sqrt{2}}=\frac{6\sqrt{2}}{4}=2{\cdot}1\text{, roughly,}$$

and its direction is given by

$$\left(\tfrac{4}{36}-\tfrac{8}{36}\right)x+\tfrac{1}{12}y=0 \quad\text{or}\quad 4x-3y=0.$$

The equation of the asymptotes is of the form

$$4x^2+12xy-y^2-40x-20y+c=0,$$

where $c$ has to be found from the condition for straight lines.

We easily find $c=60$; so the asymptotes are

$$4x^2+12xy-y^2-40x-20y+60=0.$$

Where they meet $OX$ we have $x=8{\cdot}16$ or $1{\cdot}84$;
and where they meet $OY$ $y=2{\cdot}64$ or $-22{\cdot}64$.

Marking these points, it is easy to draw the asymptotes, bearing in mind that they meet in (2, 2).

Where the curve meets $OX$ we have

$$x^2-10x+6=0;\quad\therefore\quad x=9{\cdot}3 \text{ or } {\cdot}7\text{, roughly.}$$

Similarly, where it meets $OY$

$$y^2+20y-24=0;\quad\therefore\quad y=1{\cdot}1 \text{ or } -22{\cdot}1.$$

The curve is thus as in Fig. 56.

*Example* (viii.). *Trace the curve*

$$(x-2y+1)(x+2y-3) = 5.$$

This is evidently a hyperbola, having

$$(x-2y+1) = 0, \quad (x+2y-3) = 0$$

as asymptotes.

We can decide easily in which pair of angles the curve lies by giving particular values to $x$ or $y$—*e.g.*, $x = 0$ gives $y$ imaginary and $y = 0$ gives $x = 4$ or $-2$. Or we note that for any point of the curve $x-2y+1$ and $x+2y-3$ are both positive or both negative, while for the *origin* the former is positive, the latter negative. The curve is therefore, *not* in the same angle as the origin.

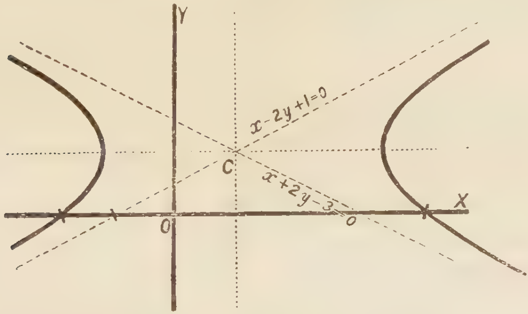



Fig. 57.

The axes bisect the angles between the asymptotes, and are therefore easily drawn. In this particular case the asymptotes $x-2y+1=0$ and $x+2y-3=0$ are equally inclined to the axes of coordinates, so that the axes of the curve are parallel to the axes of coordinates.

A few other points on the curve may be found in the ordinary way and the curve drawn.

The only defect of this method is that it does not readily lend itself to finding the actual lengths of the transverse and conjugate axes, or the eccentricity.

## MISCELLANEOUS EXERCISES ON CHAP XI.

Trace the following curves :—

1. $6x^2-xy-12y^2-8x+29y-14 = 0.$
2. $11x^2-4xy+14y^2 = 240.$
3. $9x^2-24xy+16y^2+44x+8y+5 = 0.$
4. $4(x-2y+3)^2+9(2x+y-1)^2 = 80.$

5. $5x^2+6xy-5y^2-22x+18y-7=0.$

6. $(x-2y+1)(x+2y-3)=-5.$

7. $57x^2-150xy-23y^2=34.$

8. $x^2-y^2-10x+4y-7=0.$

9. $x^2-xy+y^2-7x+8y+18=0.$

10. $(x+2y-2)^2-4(2x-y+1)^2=45.$

11. $(4x-y+12)(x-3y+9)=40.$

12. $8x^2-2xy-15y^2+6x+46y-35=0.$

13. $x^2+2x+3y+3=0.$

14. $4x^2-12xy+9y^2+8x-6y+1=0.$

15. $y^2+2xy+2y=1.$

16. $4(x-2y+3)^2-9(2x+y-1)^2=80.$

17. $y^2+4y+6x+3=0.$

18. $(4x-y+12)(x-3y+9)=108.$

19. $(x-y+1)^2+2(x+y-3)^2=0.$

20. $4x^2-12xy+9y^2+13x-26y=0.$

21. $x^2+y^2-10x+6y-2=0.$

22. $2x^2-3xy-2y^2+5y+2=0.$

23. $4x^2-12xy+9y^2-4x+6y-5=0.$

24. $(3x+4y-3)^2+(8x-6y+3)=50.$

25. $15x^2-23xy-28y^2+x+29y-6=0.$

26. $x^2+6xy+9y^2+5x+15y+12=0.$

27. $25x^2-36xy+40y^2+10x-28y-47=0.$

28. $(x+y-2)(x-y+4)+20=0.$

29. $49x^2+126xy+81y^2+23x+11y+12=0.$

30. $(x+2y)^2+3(x+2y)+2=0.$

31. $161x^2+480xy-161y^2+289=0.$

32. $7x^2-60xy+32y^2-106x+68y-37=0.$

33. $13x^2+8xy\sqrt{3}+21y^2+(75-36\sqrt{3})x+(36+75\sqrt{3})y+ .44=0.$

34. $9x^2-2\sqrt{3}xy+11y^2-2(9-\sqrt{3})x-2(11-\sqrt{3})y-2(2+\sqrt{3})=0.$

35. $x^2+3xy+4y^2-28x-56y+196=0.$

36. $3x^2+2xy+y^2-10x-14y+19=0$

37. $(x+3y)^2+(6x-2y+6)^2=40.$

## EXAMINATION PAPER III.

1. Show how to find the coordinates $(x', y')$ of the centre of the curve
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$
and prove that the asymptotes are given by the equation
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c.$$

2. A hyperbola has for its asymptotes the lines $2x - y - 3 = 0$, $3x + y - 7 = 0$, and it passes through the point $(1, 1)$. Show that it will also pass through the point $(4, \sqrt{19})$, and that its eccentricity is $\sqrt{4 + 2\sqrt{2}}$.

3. Explain, briefly, how to determine the nature of the locus represented by the general equation of the second degree, the axes of coordinates being rectangular.

4. Interpret the following equations:—

(i.) $x^2 + 6xy + 9y^2 + 4x + 12y - 12 = 0$; (ii.) $xy - 7x - 2y + 4 = 0$

Trace the following curves:—

5. $(2x - y)^2 = 5(x - 2y)$.

6. $(2x + 3y)^2 + (2x + 2y + 2) = 0$.

7. $180x^2 + 120xy + 145y^2 - 48x - 86y - 23 = 0$.

8. $25x^2 + 120xy + 144y^2 - 46x - 9y - 11 = 0$.

9. Find the equations of the two ellipses that have the lines $3x + y - 1 = 0$, $x - 3y + 2 = 0$ for their principal axes, their major and minor axes being of lengths 6 and 4 respectively.

10. Find the conditions that
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
represents two *parallel* straight lines.

# CHAPTER XII.

## CHORDS AND TANGENTS.

124. **Chords and Tangents.**—In this chapter we shall discuss some of the properties of chords and tangents, and, in all cases, we shall look on a tangent as a line which meets the curve in two coincident points, or, what is the same thing, as a line intersecting the curve in two points that are infinitely near together.

125. **A line is drawn through a given point in a given direction: to find the distances from the given point in which it is met by the conic**

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Let $(x_1, y_1)$ be the given point $O$, and suppose the direction makes an angle $\theta$ with the axis of $x$.

The coordinates of a point $P$ on the line distant $r$ from $O$ are
$$x_1 + r\cos\theta, \quad y_1 + r\sin\theta,$$
(*Cf.* Part I., § 10, B., Cor.)

and hence, if $P$ be on the curve, we must have

$$a(x_1 + r\cos\theta)^2 + 2h(x_1 + r\cos\theta)(y_1 + r\sin\theta)$$
$$+ b(y_1 + r\sin\theta)^2 + 2g(x_1 + r\cos\theta) + 2f(y_1 + r\sin\theta) + c = 0.$$

This equation contains $r$ in the second degree, and, therefore, may be looked on as a quadratic giving the required distances. It is evident *a priori* that the equation must be a quadratic, for the line meets the curve in just two points.

Arranging this equation as a quadratic in $r$, we get

$$\begin{aligned} &r^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)\\ &+2r\{(ax_1+hy_1+g)\cos\theta + (hx_1+by_1+f)\sin\theta\}\\ &+ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c = 0 \quad \ldots \; (1). \end{aligned}$$

The two roots of this equation are the required distances.

NOTE.—This proof assumes that the axes of reference are rectangular.

In the following articles we shall make some extremely important deductions from the above quadratic, which is of fundamental importance in the theory of conics.

126. **Deductions from the quadratic for $r$.—One zero root.**

The equation for $r$ has a zero root if

$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0,$$

which is just the condition that $(x_1, y_1)$ should be on the curve. This is as it should be, for only in that case can $r$ have a zero value.

127. **Two zero roots.**

The quadratic has two zero roots if $(x_1, y_1)$ is on the curve, and, in addition,

$$\cos\theta\,(ax_1+hy_1+g)+\sin\theta\,(hx_1+by_1+f)=0 \;\ldots\; (2).$$

(*Tut. Alg.*, II., § 165)

In this case the line is clearly the tangent at $(x_1, y_1)$, for it meets the curve in two points coinciding with $(x_1, y_1)$, and, therefore, the above equation gives the direction of the tangent at $(x_1, y_1)$, viz.,

$$\tan\theta = -\frac{ax_1+hy_1+g}{hx_1+by_1+f} \;\ldots\ldots\ldots\ldots\; (3).$$

**128. To find the equation of the tangent at $(x_1, y_1)$.**

[**Caution.**—A formal investigation of the equation of the tangent should embody §§ 124, 126, 127, 128.]

The equation required is

$$(y-y_1) = (x-x_1)\tan\theta, \quad \text{(Part I., § 10, B)}$$

where $\theta$ is the inclination to $OX$. By § 127 this is

$$y-y_1 = (x-x_1)\left(-\frac{ax_1+hy_1+g}{hx_1+by_1+f}\right)$$

or $$(x-x_1)(ax_1+hy_1+g)+(y-y_1)(hx_1+by_1+f) = 0.$$

The equation just found becomes, on multiplying out,

$$x(ax_1+hy_1+g)+y(hx_1+by_1+f) - (ax_1^2+2hx_1y_1+by_1^2+gx_1+fy_1) = 0.$$

But, since $(x_1, y_1)$ is on the curve

$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c = 0,$$

$$\therefore \quad ax_1^2+2hx_1y_1+by_1^2+gx_1+fy_1 = -(gx_1+fy_1+c).$$

Hence the equation of the tangent finally is

$$\boldsymbol{x(ax_1+hy_1+g)+y(hx_1+by_1+f)+gx_1+fy_1+c = 0} \quad \text{.........} \ \textbf{(4)}.$$

This may be written

$$\boldsymbol{axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c = 0},$$

which is derived from the equation of the curve by putting $xx_1$ for $x^2$, $yy_1$ for $y^2$, $xy_1+x_1y$ for $2xy$, $x+x_1$ for $2x$, and $y+y_1$ for $2y$. This form should be remembered.

**Tangents in the simple cases.**

In the parabola $y^2-4ax = 0$ the tangent is $yy_1-x(2a)-2ax_1 = 0$ by formula (4), *i.e.* $\boldsymbol{yy_1 = 2a(x+x_1)}$ ........................ **(5).**

In the ellipse $$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

the tangent is $$\boldsymbol{\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1} \quad \text{.......................} \ \textbf{(6)}.$$

In the hyperbola $$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$$

it is $$\boldsymbol{\frac{xx_1}{a^2}-\frac{yy_1}{b^2} = 1} \quad \text{.........................} \ \textbf{(7)}.$$

**129. The general method applied to the simple cases.**—To illustrate the application to particular cases, we shall investigate the "$r$" quadratic and the equation of the tangent for the parabola $y^2 = 4ax$ and the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. We start from first principles.

I. *To find the quadratic for "r" for the parabola*

$$y^2 = 4ax,$$

*and to deduce the equation of the tangent at the point* $(x_1, y_1)$.

If the line be drawn through $(x_1, y_1)$, and $r$ be one of the values required, then (by Part I., § 10, B, Cor.) the point $(x_1 + r\cos\theta,\ y_1 + r\sin\theta)$ must be on the curve, *i.e.*,

$$(y_1 + r\sin\theta)^2 = 4a\,(x_1 + r\cos\theta),$$

*i.e.*, $r^2\sin^2\theta + 2r\,\{y_1\sin\theta - 2a\cos\theta\} + y_1^2 - 4ax_1 = 0,$

the quadratic required.

One root is zero if

$$y_1^2 - 4ax_1 = 0,$$

*i.e.*, if $(x_1, y_1)$ is on the curve; another root is zero only if the line be the tangent at $(x_1, y_1)$. The condition for this is

$$y_1\sin\theta - 2a\cos\theta = 0.$$

Therefore the equation of the tangent is

$$\frac{y - y_1}{x - x_1} = \tan\theta = \frac{2a}{y_1};$$

$$\therefore \quad yy_1 - y_1^2 = 2ax - 2ax_1.$$

But $$y_1^2 = 4ax_1.$$

Therefore the equation is

$$yy_1 = 2a\,(x + x_1).$$

II. *To find the "r" quadratic for the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

*and to deduce the equation of the tangent at the point* $(x_1, y_1)$.

Here the point $(x_1+r\cos\theta,\ y_1+r\sin\theta)$ must be on the curve, and therefore

$$\frac{(x_1+r\cos\theta)^2}{a^2}+\frac{(y_1+r\sin\theta)^2}{b^2}=1,$$

*i.e.*,
$$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)+2r\left(\frac{x_1\cos\theta}{a^2}+\frac{y_1\sin\theta}{b^2}\right)$$
$$+\left(\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{b^2}-1\right)=0,$$

which is the quadratic required.

One root is zero if

$$\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{b^2}-1=0,$$

*i.e.*, if the point $(x_1, y_1)$ be on the curve; a second root can be zero only when the line drawn is the tangent at this point to the ellipse. From the quadratic the condition for this is

$$\frac{x_1\cos\theta}{a^2}+\frac{y_1\sin\theta}{b^2}=0$$

or
$$\tan\theta=-\frac{b^2x_1}{a^2y_1}.$$

Hence the equation of the tangent is

$$\frac{y-y_1}{x-x_1}=\tan\theta=-\frac{b^2x_1}{a^2y_1}$$

or
$$\frac{x_1}{a^2}(x-x_1)+\frac{y_1}{b^2}(y-y_1)=0,$$

and, since
$$\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{b^2}=1,$$

this becomes
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

### Exercises.

1. For equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ show that quadratic for $r$ is

$$r^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) + 2r\left(\frac{x_1\cos\theta}{a^2} - \frac{y_1\sin\theta}{b^2}\right) + \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 = 0.$$

2. By means of the above quadratic deduce the equation of the tangent at any point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

3. Find the equation of the tangent to the curve $x^2 + x + y = 3$ at the point (1, 1) on it.

4. Write down the equation of the tangents to the curve

$$3x^2 + 6xy + 2y^2 - 6x + 2y - 25 = 0$$

at the points where $x = -1$.

5. Find the points on the curve $x^2 + xy + y^2 = 3$ at which the tangents are parallel to the line $y = x$.

[If $(x_1, y_1)$ be such a point, the condition of parallelism gives one equation, and $x_1^2 + x_1y_1 + y_1^2 = 3$ is another equation for $x_1$ and $y_1$].

6. A straight line is drawn through a point $O$ (1, 2), making an angle 45° with $OX$, and it meets the conic $x^2 + xy + y^2 + x + y + 1 = 0$ in $P$ and $Q$. Obtain the quadratic whose roots are $OP$, $OQ$, and show that

$$OP + OQ = -\frac{11\sqrt{2}}{3}, \qquad OP \cdot OQ = \frac{22}{3}.$$

7. Show that the line through $O$ in Ex. 6 meets the conic in real points.

8. A line is drawn through the point $(-1, 0)$ making an angle $\tan^{-1}\frac{5}{12}$ with the axis of $x$. Show that the distances from $(-1, 0)$ of the points in which it meets the conic $xy + x + y = 0$ are roots of the equation $30r^2 + 156r - 169 = 0$.

What is the geometrical interpretation of the fact that one root is negative?

9. In Ex. 8 find the coordinates of the middle point of the intercepted portion.

[Find the distance from $O = \frac{1}{2}(OP + OQ)$, and then use $x' = x + r\cos\theta$, $y' = y + r\sin\theta$.]

10. **The tangent at $P$ to a parabola meets the directrix at $Q$. Prove that $PQ$ subtends a right angle at the focus.**

11. **Show that, if a straight line be drawn parallel to the axis of a parabola meeting the directrix in $M$, the curve in $O$, and the focal radius parallel to the tangent at $O$ in $Q$, then**

$$TO = OQ.$$

130. **Further deductions from the quadratic for $r$. —One infinite root.**

The quadratic for $r$ will have one root infinite if the coefficient of $r^2$ be zero, that is, if (*Tut. Alg.*, II., § 166)

$$\boldsymbol{a \cos^2\theta + 2h \sin\theta \cos\theta + b \sin^2\theta = 0} \quad \ldots\ldots \ (\mathbf{8}),$$

or
$$b \tan^2\theta + 2h \tan\theta + a = 0.$$

This may be looked on as a quadratic for $\tan\theta$, and we therefore infer that *there are two values of $\theta$ which cause the line to cut the curve at infinity.* Or, in other words, through any point two lines can be drawn each to meet the curve in one point at infinity, and they are clearly parallel to the pair of lines

$$ax^2 + 2hxy + by^2 = 0,$$

*i.e.* they are parallel to the asymptotes of the conic.

The lines are real and distinct if $ab < h^2$, *i.e.* if the conic is a hyperbola (§ 91).

They are coincident if $ab = h^2$, *i.e.* if the curve is a parabola (§ 52).

And, finally, they are imaginary when $ab > h^2$, that is when the curve is an ellipse.

All this corroborates what has been already said as to the lines meeting the three species of conics at infinity.

131. **Two infinite roots.—Equation of asymptotes.**

In order that both roots may be infinite, the coefficients of $r^2$ and $r$ must both be zero, that is, we must have

$$a \cos^2\theta + 2h \sin\theta \cos\theta + b \sin^2\theta = 0 \quad (\textit{Tut. Alg.}, \text{ II., § } 167)$$

and
$$\cos\theta\,(ax_1 + hy_1 + g) + \sin\theta\,(hx_1 + by_1 + f) = 0.$$

Thus, eliminating $\theta$ by substituting the value of $\tan\theta$ from the second equation in the first, we see that $(x_1, y_1)$ cannot be chosen arbitrarily, but must satisfy

$$b\,(ax_1 + hy_1 + g)^2 - 2h\,(ax_1 + hy_1 + g)\,(hx_1 + by_1 + f) + a\,(hx_1 + by_1 + f)^2 = 0 \ \ldots \ (\text{A}).$$

Now there can be *two* infinite roots only when the line drawn is an asymptote. Therefore $(x_1, y_1)$ lies on an asymptote if condition (A) is satisfied. We therefore infer that the equation of the two asymptotes is

$$b\,(ax+hy+g)^2 - 2h\,(ax+hy+g)\,(hx+by+f)$$
$$+a\,(hx+by+f)^2 = 0 \; \ldots \; (\text{B}).$$

We may add that, as the centre is given by

$$ax_1+hy_1+g=0 \quad \text{and} \quad hx_1+by_1+f=0,$$

the two lines $ax+hy+g=0$ and $hx+by+f=0$ represent two lines through the centre, and therefore the expression (B), which is homogeneous and of the second degree, represents also two lines through the centre, as it, of course, should, since both asymptotes pass through the centre.

The reader should now verify by actually multiplying out that this form of the equation of the asymptotes only differs from that found already (§ 110) by the constant multiplier $(ab-h^2)$.

132. **If through a point $O$ two chords are drawn in fixed directions to meet a conic in $P$, $Q$ and $P'$, $Q'$, respectively, the ratio of the rectangles $OP.OQ$ and $OP'.OQ'$ is independent of the position of $O$.**

Let $x_1$, $y_1$ be the coordinates of $O$, and suppose the chords make angles $\theta$ and $\theta'$ with the axis $OX$; then, if the conic be given by the general equation, the lengths $OP$, $OQ$ are the roots of

$$r^2\,(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)$$
$$+2r\,\{\cos\theta\,(ax_1+hy_1+g)+\sin\theta\,(hx_1+by_1+f)\}$$
$$+ax_1^{\,2}+2hx_1y_1+by_1^{\,2}+2gx_1+2fy_1+c=0. \quad (\S\ 125)$$

Hence, by the theory of quadratics,

$$OP.OQ = \frac{ax_1^{\,2}+2hx_1y_1+by_1^{\,2}+2gx_1+2fy_1+c}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta},$$

and, similarly,

$$OP'.OQ' = \frac{ax_1^{\,2}+2hx_1y_1+by_1^{\,2}+2gx_1+2fy_1+c}{a\cos^2\theta'+2h\sin\theta'\cos\theta'+b\sin^2\theta'}.$$

Consequently,

$$\frac{OP.OQ}{OP'.OQ'} = \frac{a\cos^2\theta'+2h\sin\theta'\cos\theta'+b\sin^2\theta'}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta} \;\ldots\; \mathbf{(10)}.$$

Since the value of $\dfrac{OP.OQ}{OP'.OQ'}$ does not involve $x_1$ or $y_1$, it is independent of the position of $O$, and depends *only on the directions* in which the chords are drawn.

Cor. In particular, when $P$ and $Q$ coincide with each other in $T$ and $P', Q'$ in $T'$, then the two lines are tangents, and we infer that the above ratio is the same as $\dfrac{OT^2}{OT'^2}$.

**Exercise.**

12. If the conic be a circle, deduce from the above that

$$OP.OQ = OP'.OQ'. \qquad \text{(Euc. III. 35, 36)}$$

133. **The tangents drawn from any point to a central conic are in the same ratio as the parallel semi-diameters.**

Suppose the tangents $OT$, $OT'$ from $O$ make angles $\theta$, $\theta'$ with the axis of $x$, and that $LCL'$ and $MCM'$ are the parallel diameters; then, by the general result, we have

$$\frac{OT^2}{OT'^2} = \frac{LC.CL'}{MC.CM'} = \frac{a\cos^2\theta' + 2h\sin\theta'\cos\theta' + b\sin^2\theta'}{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta'}$$

$$= \frac{CL^2}{CM^2}, \quad \text{since} \quad CL' = CL \quad \text{and} \quad CM' = CM;$$

$$\therefore \quad \frac{OT}{OT'} = \frac{CL}{CM}.$$

This result is, of course, obvious for the circle; for, in that case, the tangents are equal and the diameters are equal.

*Example.—Find the directions of the two lines through the point* (1, 1) *which meet the curve* $x^2 - 3xy + 2y^2 + 2x = 0$ *in one point at infinity, and find also the finite points in which they meet the curve.*

The distance quadratic is here

$$(1 + r\cos\theta)^2 - 3(1 + r\cos\theta)(1 + r\sin\theta) + 2(1 + r\sin\theta)^2 + 2(1 + r\cos\theta) = 0$$

or $\qquad r^2(\cos^2\theta - 3\sin\theta\cos\theta + 2\sin^2\theta) + r(\cos\theta + \sin\theta) + 2 = 0.$

If one point of intersection is at infinity,

$$\cos^2\theta - 3\sin\theta\cos\theta + 2\sin^2\theta = 0, \quad \text{whence} \quad \cot\theta = 1 \text{ or } 2.$$

The finite root is given by

$$r(\cos\theta + \sin\theta) + 2 = 0.$$

When $\cot\theta = 1$, this gives $r\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + 2 = 0$ or $r = -\sqrt{2}$.

When $\cot\theta = 2$, this gives $r\left(\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}\right) + 2 = 0$ or $r = -\frac{2\sqrt{5}}{3}$.

Thus the coordinates of the finite points of intersection

$$1 + r\cos\theta, \quad 1 + r\sin\theta$$

are for $\cot\theta = 1$, $1 - \sqrt{2}\cdot\frac{1}{\sqrt{2}}$, $1 - \sqrt{2}\cdot\frac{1}{\sqrt{2}}$, *i.e.*, $0, 0$;

for $\cot\theta = 2$, $1 - \frac{2\sqrt{5}}{3}\cdot\frac{2}{\sqrt{5}}$, $1 - \frac{2\sqrt{5}}{3}\cdot\frac{1}{\sqrt{5}}$, *i.e.*, $-\frac{1}{3}, \frac{1}{3}$.

Hence the lines make angles $45^\circ$ and $\tan^{-1}(\frac{1}{2})$ with $OX$, and the finite points of intersection are $(0, 0)$, $(-\frac{1}{3}, \frac{1}{3})$.

### Exercises.

13. Two lines are drawn through the point $(-1, 1)$, each meeting the curve $x^2 - 4xy + 3y^2 + 2x + 6y = 0$ in one point at infinity: find their directions and the finite points in which they meet the curve.

14. Show that only one line can be drawn through $(3, 2)$ to meet the curve $x^2 + 2xy + y^2 + 3x + y + 1 = 0$ in one point at infinity. Why is this? Find the finite point of intersection.

15. Find, from first principles, as in § 131, the equation of the asymptotes to

$$x^2/a^2 - y^2/b^2 = 1 \quad \text{and} \quad 4x^2 - 2xy - 3y^2 + 3x + 2y = 0.$$

16. From the result of § 132 show that, **if a central conic and a circle intersect in four points, their common chords are equally inclined to the axis of the conic.**

17. Show that, if a circle touches an ellipse at $P$ and cuts it at $Q$ and $R$, then $PQ$, $PR$, and the axis of the ellipse form an isosceles triangle.

18. Through a variable point $O$ a line is drawn in a fixed direction meeting a conic in $P$ and $Q$. Find the locus of $O$ (i.) when $OP + OQ$ is constant, (ii.) when $OP.OQ$ is constant.

What does (ii.) become for the circle?

[Take the equation of the conic to be of the general form, and use the direction-quadratic for $r$.]

134. We shall now explain another method by which the equation of the tangent at any point of a conic as well as some other important results can be deduced. This method we shall apply first of all to a simple case.

**135. To find the ratio in which the line joining two points is divided by the parabola $y^2 = 4ax$, and to deduce the equation of the tangent at the point $(x_1, y_1)$ of the parabola.**

We may remark at the outset that, as the line cuts the conic in two points, a quadratic may be expected to arise for the ratio.

Suppose $A$ $(x_1, y_1)$ and $B$ $(x_2, y_2)$ are the two given points.

If $P$ divide $AB$ so that

$$AP : PB = k : l,$$

then the coordinates of $P$ are

$$\frac{kx_2 + lx_1}{k+l}, \quad \frac{ky_2 + ly_1}{k+l}, \qquad \text{(Part I., § 3)}$$

and, accordingly, we have to determine the ratio $k : l$ so that this point may be on the curve.

Hence we have

$$\frac{(ky_2 + ly_1)^2}{(k+l)^2} = 4a\frac{kx_2 + lx_1}{(k+l)}$$

or $$(ky_2 + ly_1)^2 = 4a\,(kx_2 + lx_1)\,(k+l)\,;$$

$$\therefore\ k^2\,(y_2^{\,2} - 4ax_2) + 2kl\,(y_1y_2 - 2ax_1 - 2ax_2) + l^2\,(y_1^{\,2} - 4ax_1) = 0,$$

which is a quadratic equation in the ratio $k/l$, and gives the two values of $k/l$ in which the parabola cuts the given line.

The ratio $k/l$ has one zero value when $y_1^{\,2} - 4ax_1 = 0$, *i.e.*, when the point $A$ $(x_1, y_1)$ is on the curve. The second value of the ratio can only be zero when $(x_2, y_2)$ is on the tangent at $(x_1, y_1)$. The condition for this is

$$y_1y_2 - 2ax_1 - 2ax_2 = 0.$$

Consequently, the equation of the tangent is

$$yy_1 = 2a\,(x + x_1),$$

as found before.

**Exercise.**

Find, by the method of § 135, the tangent at any point of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

**136. To find the ratio in which the line joining two points is divided by the conic**

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

As in § 135, if $(x_1, y_1)$, $(x_2, y_2)$ are the given points and $k : l$ the required ratio, the point

$$\left(\frac{kx_2+lx_1}{k+l},\quad \frac{ky_2+ly_1}{k+l}\right)$$

must be on the curve.

The condition for this is

$$a\left(\frac{kx_2+lx_1}{k+l}\right)^2+2h\left(\frac{kx_2+lx_1}{k+l}\right)\left(\frac{ky_2+ly_1}{k+l}\right)+b\left(\frac{ky_2+ly_1}{k+l}\right)^2$$
$$+2g\left(\frac{kx_2+lx_1}{k+l}\right)+2f\left(\frac{ky_2+ly_1}{k+l}\right)+c=0,$$

or, on multiplying up by $(k+l)^2$,

$$a(kx_2+lx_1)^2+2h(kx_2+lx_1)(ky_2+ly_1)+b(ky_2+ly_1)^2$$
$$+2g(k+l)(kx_2+lx_1)+2f(k+l)(ky_2+ly_1)+c(k+l)^2=0.$$

Hence, arranging this as a homogeneous quadratic in $k$ and $l$, we get

$$k^2(ax_2^2+2hx_2y_2+by_2^2+2gx_2+2fy_2+c)+$$
$$2kl\{ax_1x_2+h(x_1y_2+x_2y_1)+by_1y_2+g(x_1+x_2)+f(y_1+y_2)+c\}$$
$$+l^2(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)=0,$$

or, as we may write it for brevity,

$$k^2S_2+2klT_{12}+l^2S_1=0 \quad\ldots\ldots\ldots\ldots\mathbf{(11)},$$

the coefficient $T_{12}$ being symmetrical in $x_1$, $y_1$ and $x_2$, $y_2$ (*i.e.*, it is not changed if we interchange $x_1$ with $x_2$ and $y_1$ with $y_2$, simultaneously).

This is clearly a quadratic for $k : l$, and therefore on being solved gives the required ratio.

This result is known as Joachimstal's.

NOTE.—Representing the general equation of the second degree by $S=0$, we shall frequently use $S_1$ to denote the result of substituting the coordinates $x_1$, $y_1$ for $x$ and $y$ in the expression $S$.

COR If the roots of the quadratic

$$\left(\frac{k}{l}\right)^2 S_2 + 2\left(\frac{k}{l}\right) T_{12} + S_1 = 0$$

are real, the line meets the curve in two real points $P$ and $Q$, and if $P$ corresponds to the value $k_1/l_1$ of the ratio, then when this is positive $P$ is between $A$ and $B$; when it is negative $P$ is outside $A$ and $B$.

If the roots are equal, the line cuts the conic in two coincident points, *i.e.*, touches it.

If the roots are imaginary, the line meets the conic in imaginary points.

We give some examples to assist the reader in appreciating the important principle just explained.

*Example* (i.). *To find the ratio in which the line joining* $(x_1, y_1)$ *to* $(x_2, y_2)$ *is divided by the line* $Ax + By + C = 0$.

If $k : l$ be the ratio required, then the point

$$\frac{kx_2 + lx_1}{k+l}, \frac{ky_2 + ly_1}{k+l}$$

is in the line, and hence we obtain

$$A(kx_2 + lx_1) + B(ky_2 + ly_1) + C(k+l) = 0$$

or

$$\frac{k}{l} = -\frac{Ax_1 + By_1 + C}{Ax_2 + By_2 + C},$$

giving the ratio required.

An important result follows at once from this formula. Suppose $(x_1, y_1)$ is $A$, and $(x_2, y_2)$ is $B$; then, when $A$ and $B$ are on opposite sides of the line, the line divides $AB$ internally; consequently the ratio $k : l$ is positive; hence $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ have opposite signs. If $A$ and $B$ are on the same side of the line, then the ratio $k : l$ is negative; so $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ have the same sign.

It follows at once that $Ax + By + C$ has the same sign for all points on one side of the line, the opposite sign for all points on the other side, while it vanishes for all points on the line. (See Part I., § 13.)

*Example* (ii.). *To find the ratio in which the line joining the points* $(\frac{15}{2}, \frac{5}{2})$ *and* $(6, 7)$ *is divided by the circle* $x^2+y^2=65$.

If the ratio be $k:l$, then the point

$$\frac{6k+\frac{15}{2}l}{k+l},\ \frac{7k+\frac{5}{2}l}{k+l}$$

is on the circle, and hence

$$(6k+\tfrac{15}{2}l)^2+(7k+\tfrac{5}{2}l)^2=65\,(k+l)^2,$$

which reduces to $$8k^2-2kl-l^2=0,$$

or, on solution, $$\frac{k}{l}=\frac{1}{2}\ \text{or}\ -\frac{1}{4}.$$

Thus one point of intersection is internal, and the other is external and nearer to the first point.

To find the actual points of intersection, we have only to use the ratio found above. Thus the coordinates of the internal point are

$$\frac{1\,.\,6+2\,.\,\frac{15}{2}}{1+2},\ \frac{1\,.\,7+2\,.\,\frac{5}{2}}{1+2}$$

or $$7,\ 4,$$

and the coordinates of the external point are

$$\frac{1\,.\,6-4\,.\,\frac{15}{2}}{1-4},\ \frac{1\,.\,7-4\,.\,\frac{5}{2}}{1-4}$$

or $$8,\ 1,$$

and it is easy to verify that these two points are actually on the curve.

## Exercises.

19. Find the ratio in which the straight line $3x+y=1$ divides the line joining the points $(-1, 1)$, $(3, 5)$.

20. Find the locus of a point such that the line joining it to $(1, 1)$ is bisected by the straight line $2x+y=5$.

21. Find the ratios in which the line joining the points $(\frac{5}{2}, \frac{7}{4})$, $(\frac{8}{3}, \frac{5}{3})$ is divided by the conic $xy-x-y=0$, and find the coordinates of the points of intersection.

22. Show that, with the notation of § 136, the line joining $(x_1, y_1)$ to $(x_2, y_2)$ meets the conic in real, coincident, or imaginary points according as $T_{12}{}^2 >=< S_1 S_2$.

23. If $O$ be a fixed point, and $P$ any point on a fixed straight line, show (without taking $O$ for origin) that, if $OP$ be divided in a given ratio at $P'$, then the locus of $P'$ is a line parallel to the locus of $P$.

[Suppose $O$ is $(a, b)$ and $P'$ is $(x, y)$; then $P$ divides $OP'$ in a fixed ratio. Write down its coordinates in terms of $a$, $b$, $x$, $y$, and express the condition that $P$ lies on a fixed line.]

**137. To find the equation of the tangent at $(x_1, y_1)$ by means of the ratio quadratic.**

The ratio quadratic is

$$k^2S+2klT_{12}+l^2S_1=0.$$

The ratio $k:l$ has one zero value when $S_1=0$, *i.e.* when the point $A$ $(x_1, y_1)$ is on the curve. This is clearly as it should be. The ratio $k:l$ has two zero roots when not only $A$ lies on the curve but $B$ is on the tangent at $A$.

The condition for this is $T_{12}=0$, *i.e.*,

$$ax_1x_2+h(x_1y_2+x_2y_1)+by_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0$$

or $x_2(ax_1+hy_1+g)+y_2(hx_1+by_1+f)+gx_1+fy_1+c=0.$

As this is the condition that $(x_2, y_2)$ should be on the tangent at $(x_1, y_1)$, it follows at once that the equation of this tangent is

$$x(ax_1+hy_1+g)+y(hx_1+by_1+f)+gx_1+fy_1+c=0 \ldots (4),$$

as we have already found by the former method.

This method has the advantage that *it applies equally well whether the axes are rectangular or oblique*, for we have never assumed any property of rectangular axes in the proof. (*Cf.* Part I., § 3.)

**Exercises.**

24. Write down the equations of the tangents to the parabola $x^2=8y$ at the points where $x=2, 3, 5$ respectively.

25. Write down the equations of the tangents to the hyperbola $x^2/9-y^2/4=1$ at the points where $y=2, 4, \frac{4}{3}$ respectively.

26. Find the equations to the tangents to the following curves at the ends of their latera recta:—

(*a*) $x^2/4+y^2=1$; (*b*) $y^2=8px$; (*c*) $y^2-9x^2=9$.

27. Find the equations to a tangent to

(*a*) $y^2=8x$, (*b*) $4x^2+9y^2=18$,

that cuts off equal lengths from the axes.

28. **Show that the equation of the tangent to the hyperbola $xy=c^2$ at the point $(x_1, y_1)$ may be written**

$$\frac{x}{x_1}+\frac{y}{y_1}=2.$$

Deduce that, if the tangent at any point of a hyperbola meet the asymptotes $CL$, $CM$ in $P$ and $Q$, then $CP \,.\, CQ$ is constant. Also $PQ$ is bisected at the point of contact.

29. Find the equation of the tangent at any point of the circle

$$x^2+2xy\cos\omega+y^2+2gx+2fy+c=0.$$

**138. To find the joint equation of the two tangents from any point to a conic.**

[Of course, for a full proof, the ratio quadratic of § 136 must be obtained and not assumed.]

If the ratio quadratic has equal roots, the line joining $A$ $(x_1, y_1)$ and $B$ $(x_2, y_2)$ touches the conic. This can only happen when $B$ lies on a tangent drawn from $A$ to the conic.

The condition for equal roots is

$$T_{12}^2 = S_1 S_2, \quad (\textit{Tut. Alg.}, \text{II.}, \S\, 159)$$

and this is therefore the condition that $(x_2, y_2)$ should be on the tangent from $A$ to the conic. As it is of the second degree in $x_2$ and $y_2$, it follows at once that there are two tangents given by the equation obtained by replacing $x_2$, $y_2$ by $x$, $y$; *i.e.* the equation is

$$\begin{aligned}\left\{axx_1+h\,(xy_1+x_1y)+byy_1+g\,(x+x_1)+f\,(y+y_1)+c\right\}^2 \\ = \quad \left\{ax^2+2hxy+by^2+2gx+2fy+c\right\} \\ \times\left\{ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c\right\} \\ \ldots\ldots\ldots\ (\mathbf{12}).\end{aligned}$$

For the simpler cases, this equation takes the following forms :—

Parabola, $y^2-4ax$; tangents from $(x_1, y_1)$ are

$$\{yy_1-2a\,(x+x_1)\}^2 = (y^2-4ax)\,(y_1^2-4ax_1).$$

Ellipse, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$; tangents from $(x_1, y_1)$ are

$$\left\{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right\}^2 = \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right).$$

Hyperbola, $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$; tangents from $(x_1, y_1)$ are

$$\left\{\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right\}^2 = \left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right).$$

The reader should obtain all these equations, starting from first principles as above.

By way of guidance, we give the investigation for the ellipse.

*Example.—To find the equation of the two tangents from the point* $P$ $(x_1, y_1)$ *to the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

If $Q$ $(x_2, y_2)$ is a point on either tangent, then the line joining this to $P$ cuts the ellipse in two coincident points. Let us find the ratios in which this line is divided by the ellipse, and put down the condition that they should be equal.

The coordinates of the point dividing $PQ$ in the ratio $k : l$ are

$$\frac{kx_2 + lx_1}{k+l}, \quad \frac{ky_2 + ly_1}{k+l}.$$

If this is on the ellipse, we have

$$\frac{(kx_2 + lx_1)^2}{(k+l)^2} \cdot \frac{1}{a^2} + \frac{(ky_2 + ly_1)^2}{(k+l)^2} \cdot \frac{1}{b^2} = 1;$$

or, on multiplying up by $(k+l)^2$ and collecting terms, we get

$$k^2\left\{\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} - 1\right\} + 2kl\left\{\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1\right\} + l^2\left\{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right\} = 0.$$

The condition that this should have equal roots in $k/l$ is

$$\left(\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1\right)^2 = \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)\left(\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} - 1\right).$$

This, therefore, is the condition that $(x_2, y_2)$ should be on one of the tangents sought; thus the equation required is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2.$$

### Exercises.

30. Find the equation of the tangents to $3x^2 + 7y^2 = 42$ from the point $(11, -5)$.

31. Find the equation of the tangents from the point $(0, p)$ to the parabola $y^2 = 4px$.

32. Find the equation of the tangents from the origin to the curve

$$(x-3a)^2/a^2 + y^2/b^2 = 1.$$

33. Find the equation of the tangents from $(a)$ $(4, 3)$, $(b)$ $(1, 1)$, to the ellipse $x^2/9 + y^2/4 = 1$. Interpret the latter result.

34. Find the equation of the tangents from $(1, 2)$ to the conic $2x^2 - 3xy - 2y^2 + x - y - 2 = 0$, and find the angle between them.

[If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of straight lines, they are parallel to the straight lines $ax^2 + 2hxy + by^2 = 0$.]

35. Show that the tangents drawn from the point $(x_1, y_1)$ to the ellipse $ax^2 + by^2 = 1$ are parallel to

$$(ax^2 + by^2)(ax_1^2 + by_1^2 - 1) = (axx_1 + byy_1)^2.$$

[Pick out the terms of the second degree.]

**139. Equation of the chord joining two points on a conic, and of the tangent at a given point.**

We shall now explain a third method of finding the equation of the tangent at any point of the curve of the second degree. This method has the advantage of giving the equation of the *chord* in a very useful form.

Let $(x_1, y_1)$ and $(x_2, y_2)$ be two points on the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Consider the equation

$$a(x-x_1)(x-x_2)+h\{(x-x_1)(y-y_2)+(x-x_2)(y-y_1)\}$$
$$+b(y-y_1)(y-y_2)$$
$$=ax^2+2hxy+by^2+2gx+2fy+c \;\dots\; \textbf{(13)}.$$

In the first place, although apparently of the second, it is really of the first, degree in $x$ and $y$ (for it is easy to see that the terms of the second degree cancel out), and therefore it is the equation of *some* straight line.

Next, if we put $x=x_1$ and $y=y_1$, the left-hand side vanishes identically, and the right-hand side vanishes, because $(x_1, y_1)$ is on the conic. Hence $(x_1, y_1)$ is on the line; similarly, $(x_2, y_2)$ is also.

Therefore (13) is actually the equation of the line joining the two points $(x_1, y_1,)$ $(x_2, y_2)$.

If we put $x_2=x_1$, $y_2=y_1$, we obtain the equation of the tangent at $(x_1, y_1)$, viz.,

$$a(x-x_1)^2+2h(x-x_1)(y-y_1)+b(y-y_1)^2$$
$$=ax^2+2hxy+by^2+2gx+2fy+c.$$

Squaring out and collecting terms, this becomes

$$2x(ax_1+hy_1+g)+2y(hx_1+by_1+f)=ax_1^{\,2}+2hx_1y_1+by_1^{\,2}-c$$
$$=-(2gx_1+2fy_1+c)-c,$$

since $(x_1, y_1)$ is on the curve.

Hence, on dividing by 2 we get the equation of the tangent as before:—

$$x(ax_1+hy_1+g)+y(hx_1+by_1+f)+gx_1+fy_1+c=0 \;\dots\; \textbf{(14)}.$$

The reader should carefully notice the mode of formation of the expression in the equation of the chord.

(*a*) On the right-hand side we place the expression which, when equated to zero, is the equation of the conic.

(*b*) On the left-hand side we place expressions depending on the *terms of the second degree only.* The rule for forming the expression on the left is this :—

*Take the terms of the second degree, replace* $x^2$ *by* $(x-x_1)(x-x_2)$, $y^2$ *by* $(y-y_1)(y-y_2)$, *one half of the* $xy$'*s by* $(x-x_1)(y-y_2)$, *and the other half by* $(x-x_2)(y-y_1)$.

The point is, of course, to form an expression which vanishes identically when $x = x_1$ and $y = y_1$, and also when $x = x_2$ and $y = y_2$, so that every term must contain either $x-x_1$ or $y-y_1$, and either $x-x_2$ or $y-y_2$ as a factor. Further, it is essential that the terms of the *second* degree should be the same on both sides.

*Examples.*—(i.) In the parabola $y^2 = 4ax$ the equation of the chord is

$$(y-y_1)(y-y_2) = y^2-4ax.$$

(ii.) In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ it is

$$\frac{(x-x_1)(x-x_2)}{a^2}+\frac{(y-y_1)(y-y_2)}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1.$$

**Exercises.**

Obtain the equations of the chords joining the points $(x_1, y_1)$ $(x_2, y_2)$ on the following curves, and deduce the equation of the tangent at $(x_1, y_1)$. Verify your answers by comparing them with the general equation.

36. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

37. $y^2 = 4ax$.

38. $xy = c^2$.

39. $ax^2+2hxy+by^2 = 1$.

40. $ax^2+2hxy+by^2+2gx = 0$.

140. **Comparison of the three methods of finding the equation to the tangent.**

The first, as we have explained it, applies only to rectangular axes, whereas the second and the third apply equally well to oblique axes. But the first has the advantage of giving the important theorems relating to the rectangle under the segments of chords which cannot be readily deduced from the second or third.

The second, depending on the ratio quadratic, enables us to find very easily the equations of the two tangents from a point to a conic.

The third method, although of less importance than the former two, gives us a simple method of writing down the equation of the chord in a form which is very often useful.

We need hardly say that all three should be remembered, and, once they are grasped, the rest of the geometry of conics will easily follow

**141. Condition that the line $lx+my+1=0$ should touch a conic.**

Various methods may be used to find this condition. As we have explained already (§ 68), we may eliminate $y$, and find the condition that the quadratic for $x$ should have equal roots. This is the best for any simple case, as it starts from first principles.

There is another method which is often useful.

Suppose the conic to be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. If $lx+my+1=0$ touches it, let $(x_1, y_1)$ be the point of contact.

As the tangent at $(x_1, y_1)$ is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$,

this line must be the same as the given one, and hence

$$\frac{x_1/a^2}{l}=\frac{y_1/b^2}{m}=\frac{1}{-1}.$$

Thus $$x_1=-a^2l, \quad y_1=-b^2m;$$

but $$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1.$$

$$\therefore \quad \boldsymbol{a^2l^2+b^2m^2=1}$$

is the required condition.

Similarly, if the conic be $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$,

the condition is $$\boldsymbol{a^2l^2-b^2m^2=1.}$$

142. In the case of the parabola $y^2=4ax$ the tangent at $(x_1, y_1)$ is

$$yy_1=2a(x+x_1),$$

and, if this be the same as $lx+my+1=0$, we must have

$$\frac{y_1}{m}=-\frac{2a}{l}=-2ax_1.$$

Hence $$x_1=-\frac{1}{l}, \quad y_1=-2a\frac{m}{l};$$

but $$y_1^2-4ax_1=0.$$

$$\therefore \quad 4a^2\frac{m^2}{l^2}+4a\frac{1}{l}=0$$

or $$\boldsymbol{am^2+l=0.}$$

**143. Alternative method for finding the condition that a line may touch a conic.**

We shall give an example of another method in a simple case, viz., to find the condition that the line $x\cos\alpha+y\sin\alpha-p=0$ should touch the ellipse $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The lines joining the origin to the points of intersection are represented by
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\left(\frac{x\cos\alpha+y\sin\alpha}{p}\right)^2=0$$
or
$$x^2\left(\frac{1}{a^2}-\frac{\cos^2\alpha}{p^2}\right)-2xy\frac{\sin\alpha\cos\alpha}{p^2}+y^2\left(\frac{1}{b^2}-\frac{\sin^2\alpha}{p^2}\right)=0.$$

If the line touches the ellipse, these two lines must coincide, *i.e.*,
$$\frac{\sin^2\alpha\cos^2\alpha}{p^4}=\left(\frac{1}{a^2}-\frac{\cos^2\alpha}{p^2}\right)\left(\frac{1}{b^2}-\frac{\sin^2\alpha}{p^2}\right),$$
which, on simplification, gives
$$p^2=a^2\cos^2\alpha+b^2\sin^2\alpha \qquad \textbf{(15).}$$

Thus the line whose equation is
$$x\cos\alpha+y\sin\alpha=\pm\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha} \qquad \textbf{(16).}$$
is a tangent to $\qquad x^2/a^2+y^2/b^2=1.$

### Exercises.

41. Find (i.) by the method of § 141, (ii.) by the method of § 143, the value of $c$ if $x+y=c$ touches the ellipse $2x^2+3y^2=4$.

42. Prove as in § 143 that
$$x\cos\alpha+y\sin\alpha=\pm\sqrt{a^2\cos^2\alpha-b^2\sin^2\alpha}$$
**is a tangent to the hyperbola** $x^2/a^2-y^2/b^2=1$.

43. Find the condition that the line $x/m+y/n=1$ should touch the ellipse $x^2/a^2+y^2/b^2=1$.

44. Find the equations of the tangents to the ellipse $x^2/2+y^2/7=1$ that makes an angle of 30° with the axis of $y$.

45. Find the distance of the origin from a tangent to $b^2x^2+a^2y^2=a^2b^2$ that makes an angle of 60° with the axis of $y$.

46. Show that the condition that $lx+my+1=0$ should touch the conic $\qquad ax^2+2hxy+by^2+2gx+2fy+c=0,$
is $\quad l^2(bc-f^2)+m^2(ca-g^2)+n^2(ab-h^2)+2mn(gh-af)$
$$+2nl(hf-bg)+2lm(fg-ch)=0.$$

**144. The locus of the point of intersection of two tangents at right angles is a circle or a straight line according as the conic is a central conic or a parabola.**

To solve this problem we write down the equation of the tangents from the point $(x_1, y_1)$ to the conic, and then expressing the condition that the pair of lines are at right angles, we obtain an equation in $x_1$ and $y_1$ which represents the locus required.

Thus, in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

the tangents from $(x_1, y_1)$ are represented by the equation

$$\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right),$$

and, if the two lines are at right angles, the sum of the coefficients of $x^2$ and $y^2$ is zero (Part I., § 29), *i.e.*,

$$\frac{x_1^2}{a^4}-\frac{1}{a^2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)+\frac{y_1^2}{b^4}-\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=0,$$

or, on simplifying,

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{x_1^2}{a^2b^2}+\frac{y_1^2}{a^2b^2}.$$

$$\therefore \quad x_1^2+y_1^2=a^2+b^2.$$

Therefore the locus required is the circle

$$\boldsymbol{x^2+y^2=a^2+b^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(17)},$$

which is concentric with the ellipse, and the square of its radius is equal to the sum of the squares of the semi-axes.

This circle is called the **director circle** of the ellipse.

In the same way, *for the hyperbola*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

the locus is the director circle

$$\boldsymbol{x^2+y^2=a^2-b^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(18)}.$$

*For the parabola* we take the equation in the form

$$y^2=4ax,$$

and now the tangents from $(x_1, y_1)$ are given by

$$\left\{yy_1-2a(x+x_1)\right\}^2=(y^2-4ax)(y_1^2-4ax_1).$$

The coefficients of $x^2$ and $y^2$ are, respectively,

$$+4a^2 \quad \text{and} \quad y_1^2-y_1^2+4ax_1.$$

The locus is therefore

$$4a^2 + y^2 - y^2 + 4ax_1 = 0$$

or

$$x + a = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(19)}.$$

This plainly represents the directrix of the curve (§ 41). Exactly the same process applies to all equations.

145. There are other useful methods of solving the problem of the last article.

For example, in the case of the ellipse we have seen (§ 68) that the line $y = mx \pm \sqrt{b^2 + a^2m^2}$ touches the ellipse for all values of $m$.

If the tangent passes through the point $(x_1, y_1)$, we have

$$y_1 - mx_1 = \pm\sqrt{b^2 + a^2m^2}$$

or

$$(y_1 - mx_1)^2 = b^2 + a^2m^2;$$

or

$$m^2(a^2 - x_1^2) + 2mx_1y_1 + b^2 - y_1^2 = 0.$$

This is a quadratic in $m$; and therefore gives the directions of the two tangents through the point $(x_1, y_1)$. If the tangents are at right angles, the product of the roots $m_1, m_2 = -1$.

$$\therefore \quad \frac{b^2 - y_1^2}{a^2 - x_1^2} = -1,$$

whence $x_1^2 + y_1^2 = a^2 + b^2$, leading to the same result as before.

We leave the reader to make similar deductions from the equation

$$y = mx + \frac{a}{m}$$

for the parabola, and from

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

for the hyperbola.

146. **Alternative method** applicable to ellipse and hyperbola.

The line $\quad x\cos\alpha + y\sin\alpha = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$

touches the ellipse $x^2/a^2 + y^2/b^2 = 1$ for all values of $\alpha$.

The tangent at right angles to this one is therefore

$$x\cos(90^\circ + \alpha) + y\sin(90^\circ + \alpha) = \sqrt{a^2\cos^2(90^\circ + \alpha) + b^2\sin^2(90^\circ + \alpha)},$$

since the perpendicular to it from the origin makes an angle $(90^\circ + \alpha)$ with the axes. This equation is

$$-x\sin\alpha + y\cos\alpha = \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}.$$

The locus of the point of intersection is obtained by eliminating $\alpha$. This is done by squaring each equation and adding. This gives at once

$$x^2 + y^2 = a^2 + b^2,$$

as before.

**147.** The equations of the type

$$y = mx + \sqrt{a^2m^2 + b^2},$$

$$y = mx + \frac{a}{m}$$

are essentially the easiest ones to use in problems where we deal with the directions of tangents. Witness, for example, the ease with which they are applied to the locus of the intersection of perpendicular tangents. In all cases we may regard $(x, y)$ as the point from which the tangents are drawn, and the equation as a quadratic giving the $m$'s of those tangents.

*E.g.*—To find the angle between the tangents drawn from $(x_1, y_1)$ to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we remark that

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2},$$

where $m_1$, $m_2$ are the roots of the quadratic

$$(y_1 - mx_1)^2 = a^2m^2 + b^2,$$

or

$$m^2(a^2 - x_1^2) + 2mx_1y_1 + b^2 - y_1^2 = 0;$$

$$\therefore \quad m_1 + m_2 = -\frac{2x_1y_1}{a^2 - x_1^2}, \quad m_1m_2 = \frac{b^2 - y_1^2}{a^2 - x_1^2}.$$

Hence

$$(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$$

$$= 4\frac{x_1^2y_1^2 - (a^2 - x_1^2)(b_2 - y_1^2)}{(a^2 - x_1)^2}$$

$$= \frac{4(b^2x_1^2 + a^2y_1^2 - a^2b^2)}{(a^2 - x_1^2)^2};$$

$$\therefore \quad \tan\theta = \frac{2\sqrt{b^2x_1^2 + a^2y_1^2 - a^2b^2}}{(a^2 - x_1^2)} \frac{(a^2 - x_1^2)}{b^2 - y_1^2 + a^2 - x_1^2}$$

$$= \frac{2ab\sqrt{\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} - 1}}{a^2 + b^2 - (x_1^2 + y_1^2)},$$

so that $\theta = 0$ only when $(x_1, y_1)$ is on the curve, in which case the tangents coincide, and $\theta = \frac{1}{2}\pi$ only when $(x_1, y_1)$ is on the director circle, as already found.

**148. In the ellipse the locus of the foot of the perpendicular from a focus on the tangents is the circle on the major axis as diameter.**

For this purpose we use the equation of the tangent in the form
$$y = mx + \sqrt{b^2 + a^2m^2},$$
the ellipse being
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
and we consider the focus $(ae, 0)$.

The equation of any line perpendicular to the tangent is
$$my + x = k,$$
and, if this pass through the focus in question,
$$m.0 + ae = k$$
or
$$k = ae.$$

Hence the foot of the perpendicular is the intersection of the lines
$$y - mx = \sqrt{b^2 + a^2m^2},$$
$$my + x = ae.$$

If we eliminate $m$, we obtain an equation satisfied by the feet of all the perpendiculars, which is, accordingly, the equation of the locus sought.

Squaring and adding the two equations, we have
$$(x^2 + y^2)(1 + m^2) = b^2 + a^2m^2 + a^2e^2$$
$$= a^2(1 + m^2), \text{ since } b^2 = a^2(1 - e^2).$$
$$\therefore \quad \boldsymbol{x^2 + y^2 = a^2} \quad \ldots\ldots\ldots\ldots \quad \textbf{(20)},$$
which proves the result required.

**Auxiliary Circle.**—DEFINITION.—The circle on the major axis as diameter is called the **auxiliary circle.**

In like manner, the feet of the perpendiculars from the other focus $(-ae, 0)$ lie on the same circle; for all that is altered is that in the equation of the perpendicular we have
$$my + x = -ae,$$
and then, on squaring, this difference disappears.

**Exercise.**

47. Apply the same method to the hyperbola, and show that the locus is now $x^2 + y^2 = a^2$, *i.e.* the circle described on the transverse axis as diameter.

149. **In the parabola the locus of the foot of the perpendicular from the focus on a tangent is the tangent at the vertex.**

Take the parabola to be $y^2 = 4ax$; then the focus is $(a, 0)$

A tangent to the parabola is

$$y = mx + \frac{a}{m}.$$

The perpendicular on this from $(a, 0)$ is

$$(x-a) + my = 0.$$

To find the locus, we have to eliminate $m$ between two equations.

Since the first is equivalent to

$$my - a = m^2 x$$

and the second to $\qquad my - a = -x,$

we have $\qquad m^2 x = -x.$

Therefore $x = 0$ is the locus required.

**150. The product of the perpendiculars from the foci of an ellipse on any tangent is equal to the square on the semi-minor axis.**

Let the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$;

then the foci are $(ae, 0)$, $(-ae, 0)$.

Now any tangent to the ellipse is

$$x \cos \alpha + y \sin \alpha = p,$$

where $\qquad p = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}.$

The perpendiculars from the foci $(-ae, 0)$, $(ae, 0)$ are, respectively, $-ae \cos \alpha - p$ and $ae \cos \alpha - p$.

Hence the product is

$$\begin{aligned} p^2 - a^2e^2 \cos^2 \alpha &= a^2 \cos^2 \alpha + b^2 \sin^2 \alpha - a^2e^2 \cos^2 \alpha \\ &= a^2 \cos^2 \alpha (1 - e^2) + b^2 \sin^2 \alpha. \end{aligned}$$

But $\qquad b^2 = a^2 (1 - e^2).$

Hence the product $= b^2 (\cos^2 \alpha + \sin^2 \alpha)$

$$= b^2.$$

**Exercises.**

48. Apply the same method to prove the proposition of § 150 for the hyperbola $$x^2/a^2 - y^2/b^2 = 1.$$

Prove the proposition for the ellipse by taking the tangent to be
$$y = mx + \sqrt{b^2 + a^2m^2}.$$

## MISCELLANEOUS EXERCISES ON CHAP. XII.

49. Find the condition that the line $lx + my = p$ may touch the conic $y^2 = 2Ax + Bx^2$.

50. A line is drawn through the point (1, 1) in a direction making an angle $\tan^{-1}(\frac{3}{4})$ with axis of $x$. Find the distances from (1, 1) in which it meets (i.) the ellipse $\frac{1}{3}x^2 + y^2 = 1$;
(ii.) the rectangular hyperbola $xy = 2$;
(iii.) the parabola $y^2 = x + 2y + 1$.

51. A line is drawn through the point $O$ $(x_1, y_1)$ in a direction making an angle $\theta$ with the axis of $X$. If it meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in $P$ and $Q$, show that
$$OP.OQ = \frac{x_1^2/a^2 + y_1^2/b^2 - 1}{\cos^2\theta/a^2 + \sin^2\theta/b^2}; \quad OP + OQ = -\frac{2(x_1\cos\theta/a^2 + y_1\sin\theta/b^2)}{\cos^2\theta/a^2 + \sin^2\theta/b^2};$$
$$\frac{1}{OP} + \frac{1}{OQ} = -\frac{2(x_1\cos\theta/a^2 + y_1\sin\theta/b^2)}{x_1^2/a^2 + y_1/b^2 - 1}.$$

52. Prove that the equation of the pair of lines which join the origin to the two points in which the straight line $x - c + \lambda y = 0$ cuts
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2x}{a} = 0 \quad \text{is} \quad x^2\left(\frac{1}{a^2} - \frac{2}{ac}\right) - \frac{2\lambda}{ac}xy + \frac{y^2}{b^2} = 0.$$

53. Find the equation of the tangent to the circle $x^2 + y^2 - 2ax = 0$, at the point $(a\overline{1+\cos\theta},\ a\sin\theta)$ on it.

54. Find the coordinates of the point of intersection of tangents drawn at the points $(x', y')$, $(x'', y'')$ to the parabola $y^2 = px$.

55. A line is drawn through the point $O$ $(x_1, y_1)$ making an angle $\theta$ with $OX$. If it meets the general conic in $P$ and $Q$, show that
$$\frac{1}{OP.OQ} = \frac{a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta}{ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c}.$$

Hence, show that, if $OPQ$, $ORS$ be two lines at right angles through $O$,
$$\frac{1}{OP.OQ} + \frac{1}{OR.OS}$$
is independent of the position of the perpendicular chords and depends only on the position of $O$.

56. Find what point of the parabola $y^2 = 4x$ is nearest to the line $y = x + 2$; also the least distance.

57. If two tangents to a parabola make angles $\theta$ and $\theta'$ with the axis, find the locus of their intersection when $\cot\theta - \cot\theta' = n$.

58. Find the inclination to the axis and the length of that chord of the conic $\quad 2x^2 + 4xy + 3y^2 + 5x - 64y + 127 = 0$ which is bisected at the point (1, 3).

59. Show that the tangents from the point $(x_1, y_1)$ to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are parallel to the pair of lines

$$(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c)(ax^2 + 2hxy + by^2) = \left\{axx_1 + h(xy_1 + x_1y) + byy_1 + gx + fy\right\}^2.$$

Write down the condition that they should be at right angles, and deduce the equation of the director circle in the form

$$(a + b)(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) = (ax + hy + g)^2 + (hx + by + f)^2.$$

60. The director circle of $x^2 + xy + y^2 + x + y = 0$ is

$$3(x^2 + y^2) + 2(x + y - 1) = 0.$$

61. When is the director circle to a hyperbola imaginary? Interpret your result.

62. Find the locus of intersection of tangents to a parabola $y^2 = 4ax$ drawn at points whose ordinates are in the ratio $p^2 : q^2$.

63. A number of ellipses on the same axis major are cut by any common ordinate. Prove that the tangents at the intersections all meet in a point.

64. Prove that the secant through $(x', y')$ and $(x'', y'')$, points on the parabola $y^2 = 4ax$, cuts the tangent at the vertex at a point whose ordinate is half the harmonic mean of $y'$ and $y''$.

65. Find the point of intersection of tangents to $xy = k^2$ at the points $(x_1, y_1)$, $(x_2, y_2)$.

66. Two tangents are drawn to the parabola $y^2 = 4ax$ from a point $(h, k)$, and make angles $\theta$, $\theta'$ with the axis. Prove that, if $k$ is constant, then $\cot\theta + \cot\theta'$ is constant.

67. Prove that the straight lines drawn from the point $(\xi, \eta)$ to meet the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at infinity are

$$a(x - \xi)^2 + 2h(x - \xi)(y - \eta) + b(y - \eta)^2 = 0.$$

Hence (or in any other way) show that the asymptotes of the hyperbola $2x^2 - xy - 6y^2 + 7y - 6 = 0$ are $x - 2y + 1 = 0$ and $2x + 3y - 2 = 0$.

68. Find the equation of the straight line which passes through the point (3, 5,) and meets the curve $4x^2 - 12xy + 9y^2 + 2x + y - 64 = 0$ at infinity.

Find also the coordinates of the other point in which this straight line meets the curve.

69. The difference of the ordinates of two points of a parabola is constant. Show that the tangents at these points intersect in a parabola having the same latus rectum.

70. Prove that the tangents to the curves $2x^2 + 3y^2 = 4$, $3x^2 - 3y^2 = 1$ at the points where the curves intersect are at right angles to each other.

71. Prove that the length of the perpendicular from the origin on the tangent at any point of the curve $x^2 + 2hxy - y^2 = c$ is inversely proportional to the distance of that point from the origin.

72. Prove that a tangent to a parabola meets the latus rectum and directrix at points equidistant from the focus.

73. Prove that the parabolas $y^2 = ax$ and $x^2 = ay$ cut one another at an angle whose tangent is $\frac{3}{4}$.

74. $P$, $Q$, $R$ are points on a parabola such that their abscissæ are in geometrical progression. Prove that the tangents at $P$ and $R$ intersect on the ordinate through $Q$.

75. Find the locus of the point of intersection of two tangents to a parabola making a given angle $\alpha$ with each other.

76. If the line joining $(x_1, y_1)$ to $(x_2, y_2)$ pass through a point common to the line $px + qy + r = 0$ and the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

prove that the equations

$$k(px_2 + qy_2 + r) + l(px_1 + qy_1 + r) = 0 \text{ and } k^2S_2 + 2klT_{12} + l^2S_1 = 0$$

have a common root.

Deduce that the equation of the lines joining $(x_1, y_1)$ to the common points is

$$\begin{aligned}(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)&(px_1 + qy_1 + r)^2 \\ &- 2(px_1 + qy_1 + r)\{x(ax_1 + by_1 + g) + y(hx_1 + by_1 + f) \\ &\qquad + gx_1 + fy_1 + c\}\{px + qy + r\} \\ &+ \{ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c\}\{px + qy + r\}^2 = 0.\end{aligned}$$

Verify this result when $(x_1, y_1)$ is the origin, by the method of § 38, Part I.

# CHAPTER XIII.

## MID-POINTS OF PARALLEL CHORDS. CONJUGATE DIAMETERS.

**151. To find the locus of the middle points of a system of parallel chords of the conic given by the general equation of the second degree.**

Suppose the chords are all parallel to $y = mx$.

The equation of the chord joining the points $(x_1, y_1)$ and $(x_2, y_2)$ on the conic is

$$a(x-x_1)(x-x_2)+h\{(x-x_1)(y-y_2)+(x-x_2)(y-y_1)\}$$
$$+b(y-y_1)(y-y_2) = ax^2+2hxy+by^2+2gx+2fy+c. \text{ (§ 139)}$$

Hence, expanding, collecting terms, and changing signs throughout, we get the equation

$$x\{a(x_1+x_2)+h(y_1+y_2)+2g\}$$
$$+y\{h(x_1+x_2)+b(y_1+y_2)+2f\}$$
$$= ax_1x_2+h(x_1y_2+x_2y_2)+by_1y_2-c.$$

This line is parallel to $y = mx$ if

$$\frac{a(x_1+x_2)+h(y_1+y_2)+2g}{h(x_1+x_2)+b(y_1+y_2)+2f} = -m. \text{ (Pt.I., § 19)}$$

But, if $(X, Y)$ be the middle point of the chord, we have

$$x_1+x_2 = 2X \text{ and } y_1+y_2 = 2Y,$$

(Part I., § 3, Cor.)

so that the above becomes

$$\frac{aX+hY+g}{hX+bY+f} = -m,$$

showing that the locus is the straight line whose equation is

$$\boldsymbol{ax+hy+g+m(hx+by+f) = 0} \text{ ......... } \mathbf{(1)}.$$

**152. Second method of finding the locus of the middle points of parallel chords.**

We use the first method of last chapter.

Let the parallel chords make an angle $\theta$ with the axis of $x$, and be parallel to $y = mx$, so that

$$m = \tan\theta.$$

Let the middle point of one of these chords be $(x_1, y_1)$. Then the distances from $(x_1, y_1)$ of the points in which it meets the conic are given by the quadratic in $r$, viz.,

$$\begin{aligned} r^2 (a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta) \\ + 2r\{\cos\theta(ax_1 + hy_1 + g) + \sin\theta(hx_1 + by_1 + f)\} \\ + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0. \end{aligned}$$

Since the chord is bisected at $(x_1, y_1)$, the distances are equal in magnitude but opposite in sign, *i.e.* their sum is zero.

The condition for this is that the coefficient of $r$ in the above quadratic should be zero, and thus we have

$$\cos\theta(ax_1 + hy_1 + g) + \sin\theta(hx_1 + by_1 + f) = 0.$$

Consequently, since $m = \tan\theta$,

$$ax_1 + hy_1 + g + m(hx_1 + by_1 + f) = 0,$$

or the locus of $(x_1, y_1)$ is the straight line

$$\boldsymbol{ax + hy + g + m(hx + by + f) = 0} \quad \ldots\ldots \textbf{(1)}.$$

Cor.—For all values of $m$, this line passes through the point of intersection of the lines

$$ax + hy + g = 0 \text{ and } hx + by + f = 0.$$

Now, in the case of a central conic, these two lines intersect in the centre, for the centre is on both of them, and hence, for a central conic, the locus in question is always a diameter of the curve. (See § 96.)

If the conic be a parabola, the lines

$$ax + hy + g = 0 \text{ and } hx + by + f = 0$$

are parallel, so that, for varying values of $m$, the loci are parallel straight lines.

We append separate investigations for the parabola $y^2-4ax=0$ and the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

I. In the parabola $y^2=4ax$ the quadratic for $r$ is

$$(y_1+r\sin\theta)^2=4a(x_1+r\cos\theta)$$

or $$r^2\sin^2\theta+2r(y_1\sin\theta-2a\cos\theta)+y_1^2-4ax_1=0.$$

If $(x_1, y_1)$ be the middle point of the chord, the roots must be equal and opposite, *i.e.*,

$$y_1\sin\theta-2a\cos\theta=0;$$
$$\therefore\quad y_1=2a\cot\theta;$$

Therefore for a system of chords making an angle $\theta$ with the axis, the locus of the middle points is the line

$$y=2a\cot\theta.$$

II. In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ the quadratic for $r$ is

$$\frac{(x_1+r\cos\theta)^2}{a^2}+\frac{(y_1+r\sin\theta)^2}{b^2}=1$$

or $$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)+2r\left(\frac{x_1\cos\theta}{a^2}+\frac{y_1\sin\theta}{b^2}\right)$$
$$+\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0.$$

If $(x_1, y_1)$ be the middle point of the chord, the coefficient of $r$ must be zero or

$$\frac{x_1\cos\theta}{a^2}+\frac{y_1\sin\theta}{b^2}=0;$$

Therefore the locus of the middle points of the chords, making an angle $\theta$ with the major axis, is the line

$$\frac{x\cos\theta}{a^2}+\frac{y\sin\theta}{b^2}=0.$$

If the lines are parallel to $y=mx$, we have $\tan\theta=m$, and the locus is

$$y=-\frac{b^2}{a^2}\frac{\cos\theta}{\sin\theta}x \quad\text{or}\quad y=-\frac{b^2}{a^2m}x.$$

153. **Only one chord of a conic can be drawn which is bisected at a given point.**

The equation of a chord through $(x_1, y_1)$ is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta},$$

but when the chord is bisected at $(x_1, y_1)$, we have

$$\tan\theta=-\frac{ax_1+hy_1+g}{hx_1+by_1+f}; \qquad (\S\ 151)$$

so that there is only one direction for the line, and therefore there is only one such chord, and its equation is

$$y-y_1=(x-x_1)\tan\theta,$$

when $\tan\theta$ has the value given; that is, the equation is

$$\boldsymbol{(x-x_1)(ax_1+hy_1+g)+(y-y_1)(hx_1+by_1+f)=0} \quad \text{..(2)}.$$

This form of the equation of a chord is often useful.

For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we find at once that it is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2},$$

and for the parabola $y^2=4ax$ it is

$$yy_1-2ax=y_1^2-2ax_1.$$

The general rule for writing down the equation is: "Put down the left-hand side as though the equation of the tangent was required, and then on the right-hand side place an absolute term so that the line represented passes through $(x_1, y_1)$."

For an illustration of its application see Ex. 1, p. 187.

**Exercises.**

1. Find, from first principles, by both methods (§§ 151, 152), the locus of the middle points of the chords of $x^2+xy+x+y=0$ which are parallel to (i.) $y=x$, (ii.) $y=0$.

2. Show that in the parabola $y^2=4ax$ the locus of the middle points of chords parallel to $y=mx$ is the straight line $y=2a/m$.

3. In the hyperbola $x^2/a^2-y^2/b^2=1$ use both methods to show that the locus is the straight line $y=b^2x/a^2m$.

4. In the general case the locus of the middle points of all chords which pass through the fixed point $(\xi, \eta)$ is the conic

$$ax^2+2hxy+by^2+x(g-a\xi-h\eta)+y(f-h\xi-b\eta)-\xi g-\eta f=0.$$

[Put down the condition that chord bisected at $(x, y)$ passes through $(\xi, \eta)$.]

154. Having explained the methods which may be applied to the general equation, we shall now discuss the properties of systems of parallel chords in greater detail for the ellipse, hyperbola, and parabola. It will be seen that, although there is great similarity between the properties deduced for the ellipse and hyperbola, there are, nevertheless, important differences, just as, although the equations are so similar when referred to the principal semi-axes as axes of coordinates, the forms of the curves are widely different, and (what is of importance in what follows) one of the axes does not meet the hyperbola in real points. The properties for the parabola, as far as we can treat them in this book, are different from those for the central conics.

155. **In a central conic, if the first of two diameters, bisect all chords parallel to the second, then the second will bisect all chords parallel to the first.**

Take the centre as origin; then the equation of the conic is of the form $ax^2+2hxy+by^2=1.$

Let $y=mx$, $y=m_1x$ be the two diameters.

Let $(x_1, y_1)$, $(x_2, y_2)$ be the extremities of a chord parallel to $y=mx$.

As in § 151, the equation of the chord is

$$a(x-x_1)(x-x_2)+h\{(x-x_1)(y-y_2)+(x-x_2)(y-y_1)\}$$
$$+b(y-y_1)(y-y_2)=ax^2+2hxy+by^2-1,$$

and, as it is parallel to $y=mx$, we have

$$\frac{\text{coeff. of } x}{\text{coeff. of } y}=-m$$

or
$$\frac{a(x_1+x_2)+h(y_1+y_2)}{h(x_1+x_2)+b(y_1+y_2)}=-m,$$

so that, if $(x, y)$ be the middle point of the chord, we have

$$\frac{a\,.\,2x+h\,.\,2y}{h\,.\,2x+b\,.\,2y}=-m.$$

Thus the locus of the middle points of chords parallel to $y=mx$ is

$$ax+hy+m(hx+by)=0 \quad \text{or} \quad x(a+hm)+y(h+bm)=0.$$

By hypothesis this is $y = m_1 x$;

$$\therefore \quad m_1 = -\frac{a + hm}{h + bm}$$

or

$$a + h(m + m_1) + bmm_1 = 0 \quad \dots\dots\dots (4).$$

Similarly, the condition that $y = mx$ bisects chords parallel to $y = m_1 x$ is

$$bm_1 m + h(m_1 + m) + a = 0.$$

But this is the same condition as before. Hence the proposition is proved. The above condition should be remembered.

Note.—The equation of the diameter bisecting chords parallel to $y = mx$ might have been written down as a particular case of the general result, viz., $g = f = 0$ and $c = -1$; but it is better to repeat the work, so that only the equation of the chord has to be remembered.

156. **Conjugate diameters.**—Definition.—When two diameters are such that each bisects all chords parallel to the other they are said to be conjugate.

As a simple case of conjugate diameters we may mention two perpendicular diameters of a circle, for each clearly bisects all chords parallel to the other.

157. **The tangents at the ends of a diameter are parallel to the conjugate diameter.**

Suppose $Q_1Q_1'$, $Q_2Q_2'$, $Q_3Q_3'$ (Fig. 58) are a system of parallel chords, and $CP$ the diameter bisecting them; then we have to show that the tangent at $P$ is parallel to each of the chords.

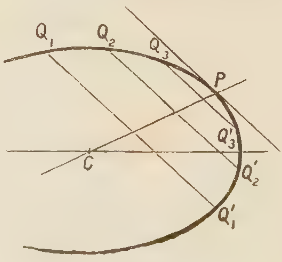


Fig. 58.

Now $Q_1$ and $Q_1'$, $Q_2$ and $Q_2'$, ... are on opposite sides of $CP$ always, but when the extremities of the chord are infinitely close together it becomes a tangent, and, since its ends are on opposite sides of $CP$, when they coincide they do so at $P$. Thus the tangent at $P$ is the ultimate position of one of the chords, and hence it is parallel to $Q_1Q_1'$, &c.

## Exercises.

5. The equation of the chord of the ellipse $x^2/a^2 + y^2/b^2 = 1$ which is bisected at $(x_1, y_1)$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$.

6. **Show, from first principles as in § 155, that, if $y = mx$ bisects all chords of $x^2/a^2 + y^2/b^2 = 1$ parallel to $y = m'x$, then**
$$mm' = -b^2/a^2.$$

7. In the case of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ show that it is in like manner
$$mm' = b^2/a^2.$$

8. Prove the truth of § 157 by analysis.

[Take the conic to be $ax^2 + 2hxy + by^2 = 1$. The tangent at $(x_1, y_1$ is
$$x(ax_1 + hy_1) + y(hx_1 + by_1) = 1.$$
This is parallel to $y = mx$ if $(ax_1 + hy_1) + m(hx_1 + by_1) = 0$, *i.e.*, if $(x_1, y_1)$ is on the conjugate diameter.]

9. Write down the condition that the lines $y = mx$, $y = m_1x$ should be conjugate diameters of $x^2 + xy + y^2 = 1$.

10. Show that the asymptotes of
$$ax^2 + 2hxy + by^2 = 1$$
are conjugate diameters of
$$Ax^2 + 2Hxy + By^2 = 1,$$
if
$$aB + bA - 2hH = 0.$$

11. Show that, if the equation to an ellipse be $2x^2 + 3y^2 = 4$, the diameters $y = 2x$ and $x + 3y = 0$ are conjugate.

12. Using the result of Ex. 6, find the condition that the pair of lines represented by $Ax^2 + 2Hxy + By^2 = 0$ should be conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

13. **Show that the condition that the pair of lines represented by $Ax^2 + 2Hxy + By^2 = 0$ should be conjugate diameters of the conic $ax^2 + 2hxy + by^2 = 1$ is**
$$aB + bA - 2hH = 0.$$

[See § 155. This result should be remembered.]

14. **In a rectangular hyperbola conjugate diameters are equally inclined to either asymptote.**

[Take equation to be $xy = c^2$.]

15. What are conjugate diameters of a circle if the circle be considered as a case of central conics? Show from the formulæ of this chapter that they are at right angles.

158. **Properties of conjugate diameters of the ellipse.**

Suppose $PCP'$ and $DCD'$ are conjugate diameters of the ellipse $x^2/a^2+y^2/b^2=1$, $a$ and $b$ being the semi-axes; let $P$ be $(x_1, y_1)$ and $D$ $(x_2, y_2)$, so that

$$P' \text{ is } (-x_1, -y_1)$$
and $$D' \text{ is } (-x_2, -y_2).$$

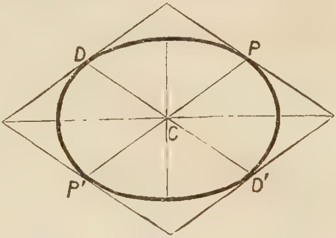


Fig. 59.

I. $$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=0 \;\ldots\; (5).$$

For let the equations of $CP$ and $CD$ be $y=m_1x$ and $y=m_2x$; then, since they are conjugate diameters, $m_1m_2=-b^2/a^2$. (§ 157 and Ex. 6)

But $$m_1=y_1/x_1 \quad \text{and} \quad m_2=y_2/x_2;$$

so that $$\frac{y_1y_2}{x_1x_2}=-\frac{b^2}{a^2} \quad \text{or} \quad \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=0.$$

II. $$\frac{x_1}{a}=\pm\frac{y_2}{b} \quad \text{and} \quad \frac{y_1}{b}=\mp\frac{x_2}{a} \quad \ldots\ldots\ldots\ldots\ldots (6);$$

either the two lower or the two upper signs being taken.

For by I. we have $$\frac{x_1/a}{y_2/b}=-\frac{y_1/b}{x_2/a},$$

and, by the properties of proportion, each of these

$$=\frac{\sqrt{x_1^2/a^2+y_1^2/b^2}}{\sqrt{x_2^2/a^2+y_2^2/b^2}}=\frac{\sqrt{1}}{\sqrt{1}}=\pm 1,$$

The result follows at once. The reason of the ambiguous sign is that when we take $P$ to be a definite end of the one diameter, $D$ may be either end of the other. In our figure the upper sign refers to $D$ and the lower one to $D'$.

Cor. $$y_2=\pm\frac{b}{a}x_1; \qquad x_2=\mp\frac{a}{b}y_1.$$

III. **The sum of the squares on two conjugate semi-diameters is equal to the sum of the squares on the semi-axes** ............................................................ (7).

For $$CP^2=x_1^2+y_1^2 \quad \text{and} \quad CD^2=x_2^2+y_2^2,$$

but $$x_2^2=\frac{a^2}{b^2}y_1^2 \quad \text{and} \quad y_2^2=\frac{b^2}{a^2}x_1^2; \qquad \text{(by II.)}$$

hence $$CP^2+CD^2=x_1^2+y_1^2+\frac{a^2}{b^2}y_1^2+\frac{b^2}{a^2}x_1^2$$

$$=x_1^2\left(1+\frac{b^2}{a^2}\right)+y_1^2\left(1+\frac{a^2}{b^2}\right)=\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\right)(a^2+b^2).$$

$$\therefore\; CP^2+CD^2=a^2+b^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

IV. **The parallelogram constructed on $CP$ and $CD$ as adjacent sides is equal to $ab$** .................................... **(8).**

The parallelogram $= 2\triangle CPD = 2 \cdot \frac{1}{2}(x_1y_2 - x_2y_1) = x_1y_2 - x_2y_1$

$$= x_1 \frac{b}{a} x_1 + y_1 \frac{a}{b} y_1 = ab\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) \quad \text{(by II.)}$$

$= ab$, as required.

V. $$CD^2 = SP.S'P.$$

We express both in terms of $x_1$, the abscissa of $P$. Thus

$$CD^2 = x_2^2 + y_2^2 = \frac{a^2}{b^2} y_1^2 + \frac{b^2}{a^2} x_1^2 = a^2\left(1 - \frac{x_1^2}{a^2}\right) + \frac{b^2}{a^2} x_1^2$$

$$= a^2 - \frac{a^2 - b^2}{a^2} x_1^2 = a^2 - e^2x_1^2. \qquad (\S\ 55)$$

Again, $$SP.S'P = (a + ex_1)(a - ex_1) = a^2 - e^2x_1^2. \qquad (\S\ 63)$$

$$\therefore \quad CD^2 = SP.S'P \text{ ............................ } \mathbf{(9)}.$$

VI. **If $p$ be the perpendicular from the centre on the tangent at $P$, then** $p.CD = ab.$

The tangent at $P$ $(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

$$\therefore \quad p = \frac{1}{\sqrt{x_1^2/a^4 + y_1^2/b^4}},$$

$$CD^2 = x_2^2 + y_2^2 = \frac{a^2}{b^2} y_1^2 + \frac{b^2}{a^2} x_1^2 = a^2b^2\left(\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}\right).$$

Hence $$p.CD = ab \text{ ......................... } \mathbf{(10)}.$$

Or VI. can be obtained at once from IV. by noticing that the tangents at $P$ and $D$ form with $CP$ and $CD$ the parallelogram of IV.

159. **Equi-conjugate diameters.**

If two diameters of an ellipse be conjugate and equal, they are called the equi-conjugate diameters.

We leave their properties as exercises for the reader as follows :—

### Exercises.

16. **In an ellipse the tangents at the ends of the axes form a rectangle. Show that its diagonals are equi-conjugate diameters,** and find their equations.

17. If $CP$, $CD$ are the equi-conjugate diameters, show that

(i.) $CP^2 = CD^2 = \frac{1}{2}(a^2 + b^2)$; (ii.) $\angle PCD = 2\tan^{-1}(b/a)$.

18. Prove that the product of two conjugate diameters is greatest when they are equal.

19. Show that the point $P$ (1, 2) is on the ellipse $2x^2+3y^2=14$.
Find the length (i.) of $CP$, (ii.) of the conjugate semi-diameter $CD$, (iii.) of the perpendicular from the centre on the tangent at $P$.
Find also the focal distances of $P$ and the angle between $CP$ and $CD$, and hence verify the properties III., IV., V., VI. of § 158.

20. Verify that the point $P$ (1, 1) is on the ellipse $x^2+xy+y^2=3$, and find the coordinates of the extremities of the diameter conjugate to $CP$.

21. **If $\alpha$, $\beta$ be two semi-conjugate diameters and $\omega$ the angle between them, show that**

$$\alpha^2+\beta^2=a^2+b^2, \quad \alpha\beta\sin\omega=ab.$$

Hence, being given two conjugate diameters in magnitude and position, show how to find the lengths of the semi-axes.

22. If, in Ex. 21, $\alpha=2$, $\beta=1$, $\omega=30°$, find $a$ and $b$ to two places of decimals.

23. If the equi-conjugate semi-diameters be each 3 units in length, and the angle between them 45°, find the semi-axes correct to two places of decimals.

160. **Equation of the ellipse referred to two conjugate diameters as axes.**

The equation must be of the form

$$Ax^2+2Hxy+By^2=1,$$

since the centre is the origin (§ 94). But the line $y=0$ bisects all chords parallel to the axis of $y$, so that any value of $x$ must give two equal and opposite values of $y$ (*i.e.*, in the quadratic for $y$ in terms of $x$ the coefficient of $y$ must be zero); hence $H=0$, and equation is of form $$Ax^2+By^2=1.$$

Again, let $2a'$ and $2b'$ be the lengths of the diameters; then in the equation $$Ax^2+By^2=1$$
$y=0$ must make $$x=\pm a';$$

$$\therefore \quad A=\frac{1}{a'^2}; \quad \text{similarly,} \quad B=\frac{1}{b'^2}.$$

Thus the equation is $$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1 \quad \text{.........................} \quad \mathbf{(11)},$$

an equation of exactly the same form as when the principal axes are taken as axes; but it must be borne in mind that the axes are now oblique.

NOTE.—The equation of the tangent, of the diameter bisecting chords parallel to $y=mx$, and the condition that two diameters should be conjugate are just the same as with rectangular axes.

**Exercise.**

24. **The equation of an ellipse referred to its equi-conjugates as axes is of the form** $x^2+y^2=c^2$.

161. **Conjugate diameters of a hyperbola.—If one of a pair of conjugate diameters of a hyperbola meets the curve in real points, the other meets it in imaginary points.**

Suppose the hyperbola is

$$x^2/a^2 - y^2/b^2 = 1\,;$$

then the diameter $y = mx$ only meets the curve in real points if $m$ is numerically less than $b/a$, for $y = bx/a$ is an asymptote, and no line further away from the transverse axis than an asymptote meets the curve (§ 76).

Now the lines $y = m_1 x$ and $y = m_2 x$ are conjugate diameters of the hyperbola if

$$m_1 m_2 = b^2/a^2,$$

so that, if $m_1 < b/a$, $m_2 > b/a$, and, if $m_1 > b/a$, $m_2 < b/a$, and hence one of the two meets the curve in real points and the other in imaginary points.

NOTE.—A system of parallel chords may be such that the extremities of any chord of the system lie on the same branch of the curve or on different branches. In the first case, we can clearly move the chord parallel to itself so that it becomes a tangent, and hence the conjugate diameter which passes through the point of contact of the tangent meets the curve in real points, one on each branch. But, if the ends lie on different branches, the chord can never become very small because the branches never come close together, and hence the conjugate diameter does not meet the curve in real points.

162. In the ellipse we defined the ends of a diameter conjugate to a given one to be the points in which it meets the curve. Since, however, in the hyperbola one of two conjugate diameters is certain to have imaginary extremities (§ 161), we cannot define them in the same way, but must introduce some entirely new considerations. These we shall explain in the next few articles.

163. **Conjugate hyperbola.**—DEFINITION.—The hyperbola whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad \text{.................} \quad (12)$$

is called the **hyperbola conjugate** to that whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = +1.$$

**164. Properties of a hyperbola and the conjugate hyperbola.**

I. **The two curves have their axes in the same directions, but the transverse axis of the one is the conjugate axis of the other, and *vice versa*.**

In fact, bearing in mind that the transverse axis is that which meets the curve in real points, we see that $y = 0$ is the transverse axis of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = +1,$$

while it is the conjugate axis of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1,$$

since it meets the latter in imaginary points. Clearly, $x = 0$ is the transverse axis of the latter.

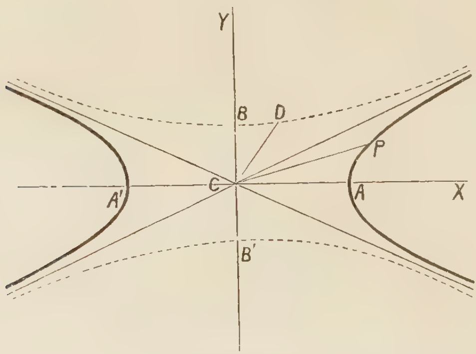



Fig. 60.

II. **The two curves have the same asymptotes.**

The asymptotes of both are clearly

$$\frac{x}{a} = \pm \frac{y}{b},$$

for to obtain them we equate to zero the terms of the second degree. (§ 86.)

III. **Any line through the common centre meets one in real and the other in imaginary points, unless it be one of the common asymptotes.**

For, where the line $y = mx$ meets the first we have

$$x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) = +1;$$

and where it meets the second

$$x^2\left(\frac{1}{a^2} - \frac{m^2}{b^2}\right) = -1.$$

Thus the first is met in real points, *i.e.* the value of $x^2$ is positive if $\quad \dfrac{1}{a^2} - \dfrac{m^2}{b^2}\quad$ is positive,

and the second is met in real points if

$$\frac{1}{a^2} - \frac{m^2}{b^2} \text{ is negative};$$

and hence, unless $\qquad m = \pm b/a,$

one curve is met in real and the other in imaginary points.

The figures of the two curves are thus as shown, the continuous one being

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = +1,$$

and the dotted one being

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

IV. **If a pair of diameters are conjugate with respect to one hyperbola, they are conjugate with respect to the conjugate hyperbola.**

The lines $y = m_1x$ and $y = m_2x$ are conjugate with respect to $\qquad x^2/a^2 - y^2/b^2 = 1$

if $\qquad m_1m_2 = b^2/a^2.$

We deduce the corresponding condition for conjugacy to $\qquad x^2/a^2 - y^2/b^2 = -1 \quad \text{or} \quad y^2/b^2 - x^2/a^2 = 1,$

by interchanging $x$ and $y$, $a$ and $b$, and writing $1/m_1$ for $m_1$ and $1/m_2$ for $m_2$ [since $y = m_1x$ makes an angle whose tangent is $1/m_1$ with the axis of $y$].

The requisite condition is therefore

$$1/m_1 \times 1/m_2 = a^2/b^2 \quad \text{or} \quad m_1 m_2 = b^2/a^2,$$

which is the same as before. Hence the proposition follows.

We see at once that, of a pair of conjugate diameters to a hyperbola, one meets the hyperbola in real points and the other meets its conjugate in real points.

Hence we make a new definition, as follows :—

165. **Conjugate semi-diameter.**—DEFINITION.—If a diameter $PCP'$ meet the hyperbola $x^2/a^2 - y^2/b^2 = 1$ in the real points $P$, $P'$, then the conjugate diameter meets the conjugate hyperbola $x^2/a^2 - y^2/b^2 = -1$ in real points $D$, $D'$, and $CD$ is called the semi-diameter conjugate to $CP$ in magnitude and direction.

Thus the extremities of the conjugate diameter are the points in which it meets the conjugate hyperbola.

**166. To find the equation of the hyperbola conjugate to a given hyperbola.**

In the simple form, the equations of the hyperbola, asymptotes, and conjugate hyperbola are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

Now to transform to any other axes, oblique or rectangular, we use substitutions of the form

$$x = l_1 x + m_1 y + n_1, \quad y = l_2 x + m_2 y + n_2.$$

(Pt.I.,§35)

Thus the three equations become

$$\frac{(l_1 x + m_1 y + n_1)^2}{a^2} - \frac{(l_2 x + m_2 y + n_2)^2}{b^2} = 1,$$

$$\frac{(l_1 x + m_1 y + n_1)^2}{a^2} - \frac{(l_2 x + m_2 y + n_2)^2}{b^2} = 0,$$

$$\frac{(l_1 x + m_1 y + n_1)^2}{a^2} - \frac{(l_2 x + m_2 y + n_2)^2}{b^2} = -1.$$

We infer that, *to form the equation of the asymptotes of a hyperbola, we have to subtract a certain quantity, say* λ, *from the absolute term, and then, to find the equation of the conjugate hyperbola, we have to subtract the same quantity* λ *from the equation of the asymptotes.* Note that λ is not necessarily unity, for the equations may be each multiplied by the same quantity. (See § 90.)

*Example.—To find the equation of the hyperbola conjugate to*

$$xy + x + y - 4 = 0.$$

This equation may be written

$$(x+1)(y+1) - 5 = 0.$$

The asymptotes must be $(x+1)(y+1) = 0$.

Hence the conjugate curve is

$$(x+1)(y+1) + 5 = 0 \quad \text{or} \quad xy + x + y + 6 = 0.$$

### Exercises.

25. Find the equations of the hyperbolas conjugate to
$\frac{1}{2}x^2 - y^2 = 1$, $3x^2 - y^2 = 2$, $Ax^2 + 2Hxy + By^2 = C$, $xy + 2x + 3y + 1 = 0$.

26. Find the equation of the asymptotes of

$$x^2 + 2xy - y^2 + 2x + 4y = 0,$$

and deduce the equation of the conjugate hyperbola.

27. If $CP$, $CD$ be two conjugate diameters of a hyperbola, $D$ being on the conjugate hyperbola, prove that the tangent at $D$ is parallel to $CP$.

167. **Properties of conjugate diameters of a hyperbola.** Suppose $(x_1, y_1)$ and $(x_2, y_2)$ are the extremities of two conjugate semi-diameters $CP$, $CD$ of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

$P$ being on this hyperbola, and therefore $D$ on the conjugate hyperbola, so that we have

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \quad \text{and} \quad \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} = -1.$$

The proofs of the following properties are almost identical with those of the ellipse (§ 158), and should be worked as an exercise by the reader.

(i.) $$\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} = 0 \quad \text{.........................} \quad (13).$$

(ii.) $$\frac{x_1}{a} = \pm \frac{y_2}{b} \quad \textit{and} \quad \frac{y_1}{b} = \pm \frac{x_2}{a} \quad \text{................} \quad (14).$$

(iii.) $$CP^2 - CD^2 = a^2 - b^2 \quad \text{.........................} \quad (15).$$

(iv.) *The parallelogram constructed on $CP$ and $CD$ as adjacent sides is equal to $ab$* ........................................................ (16).

(v.) $$CD^2 = SP \,.\, S'P \quad \text{.........................} \quad (17).$$

(vi.) *If $p$ be the perpendicular on the tangent at $P$, then $p \,.\, CD = ab$* ......... (18).

168. **Equation of the hyperbola referred to two conjugate diameters as axes.**

Since the centre is the origin, the equation is of the form

$$Ax^2+2Hxy+By^2 = 1.$$

Now any line parallel to the axis of $y$ is bisected by the axis of $x$.

Therefore any value of $x$ gives two equal and opposite values of $y$, and therefore $H=0$.

Hence the equation becomes of the form

$$Ax^2+By^2 = 1.$$

The equation of the asymptotes, which are two straight lines through the origin, is therefore

$$Ax^2+By^2 = 0.$$

Therefore, by § 166, the equation of the conjugate hyperbola is

$$Ax^2+By^2 = -1.$$

Now suppose that, of the two diameters, the one along the axis of $x$ is of length $2a'$ and meets this hyperbola, while the second is of length $2b'$ and meets the conjugate hyperbola. Then, when we make $y=0$ in $Ax^2+By^2=1$, we must get

$$x = \pm a';$$

and when we make $x=0$ in $Ax^2+By^2=-1$, we must get

$$y = \pm b'.$$

Hence we have $\quad Aa'^2 = 1 \quad$ and $\quad Bb'^2 = -1,$

so that the equation is $\quad \dfrac{x^2}{a'^2}-\dfrac{y^2}{b'^2}=1$ ........................ **(19).**

## Exercises.

28. Find the diameter of the hyperbola $x^2+xy-y^2=1$ which is conjugate to $x+2y=0$, and actually verify that, of the two diameters, one meets the curve in real points, and the other in imaginary points.

29. Show that, if the parallelogram $CPLD$ be completed, then $L$ lies on one of the asymptotes.

30. **If $\alpha$, $\beta$ be a pair of conjugate semi-diameters of a hyperbola, and $\omega$ the angle between them, show that**

$$\alpha^2-\beta^2 = a^2-b^2 \quad \textbf{and} \quad \alpha\beta\sin\omega = ab.$$

Hence, being given a pair of conjugate diameters and the angle between, show how to find the lengths of the axes.

31. In Ex. 30, if $\alpha=\sqrt{3}$, $\beta=1$, $\omega=30°$, find $a$ and $b$ to two places of decimals.

32. The point $P$ (1, 1) being on the hyperbola $3x^2-2y^2=1$, with the usual notation find the lengths of $CP$, $CD$, and the angle between $CP$ and $CD$.

Verify that $SP\,.\,S'P = CD^2$ by actually finding $SP$ and $S'P$.

**169. Parallel chords of a parabola.—The locus of the middle points of a system of parallel chords of a parabola is a straight line parallel to the axis of the curve.**

We may use the general result, but it is as easy to work *ab initio*. Take the parabola to be $y^2 = 4ax$, the axes being rectangular.

The chord joining the points $(x_1, y_1)$, $(x_2, y_2)$ on the curve is

$$(y-y_1)(y-y_2) = y^2-4ax$$

or

$$y(y_1+y_2)-4ax = y_1y_2.$$

If this be parallel to $y = mx$, we have

$$\frac{4a}{y_1+y_2} = m,$$

Since $\frac{1}{2}(y_1+y_2)$ is the ordinate of the middle point of the chord, that point lies on the line

$$\frac{y}{2a} = \frac{1}{m} \quad \text{or} \quad y = \frac{2a}{m},$$

which establishes the result.

**170. Diameter of parabola.**—Definition.—A line parallel to the axis of a parabola is called a diameter of the curve.

Note that, whereas the diameters of a central conic all meet in a point, those of a parabola are all parallel to each other. To make the analogy complete we suppose the parallel lines to meet in an infinitely distant point, the centre of the parabola.

171. **A diameter bisects all chords parallel to the tangent at its extremity.**

The proof, which is precisely as in the ellipse, we leave as an exercise for the reader.

**Exercises.**

33. In the parabola $y^2 = 3x$, find the diameter conjugate to chords parallel to $y = x$, and find also the coordinates of its extremity.

34. In the parabola $y^2 = 4x+3$, find the diameter which bisects chords parallel to $2x-y = 0$, find the coordinates of its extremity, and verify that the tangent there is actually parallel to the system of chords.

**172. Equation of a parabola when a diameter and the tangent at its extremity are taken as axes.**

Let the diameter be $OX$ and the tangent $OY$.

To find to what the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

reduces in this case notice that—

(i.) The origin is on the curve;

(ii.) Any value of $x$ should give two equal and opposite values of $y$;

(iii.) Any line parallel to $OX$ meets the curve in one point at infinity. (§ 49.)

From (i.) we have

$$c=0.$$

From (ii.)

$$h=0, \quad f=0,$$

so that the equation reduces to

$$ax^2+by^2+2gx=0.$$

From (iii.), when $y=0$, the values of $x$ have to be 0 and $\infty$; hence $a=0$ (*Tut. Alg.*, II., § 166), and the equation is

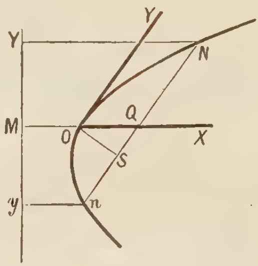


Fig. 61.

$$by^2+2gx=0,$$

which may obviously be written in the form

$$y^2=4a'x.$$

(The fact that $a=0$ follows also because the terms of the second degree form a square, and so $ab=0$ and $b\neq 0$, for then the curve would be two straight lines.)

*The value of $a'$ in the equation* $y^2=4a'x$ can easily be proved to be the focal distance of the point of contact, *i.e.*, $SO$.

Draw the focal chord $NQSn$ parallel to $OY$, meeting the curve in $N$ and $n$, and the axis of $X$ in $Q$.

Draw $NY$, $ny$ perpendicular to the directrix, and let $OX$ meet the directrix in $M$.

Then
$$NQ^2=4a'OQ.$$

But
$$NQ=\tfrac{1}{2}Nn=\tfrac{1}{2}(SN+Sn)=\tfrac{1}{2}(NY+ny)$$
$$=QM=2OM \quad \text{(by Ex. 11, p. 145)}$$
$$=2SO.$$

$$\therefore \quad 4SO^2=4a'.OQ=4a'.OM=4a'.SO.$$

$$\therefore \quad a'=SO.$$

173. We see, by comparing the results of §§ 160, 168, 172 with the equations of central conics referred to the principal diameters, that the *form* of the equations

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1,\quad \frac{x^2}{a'^2}-\frac{y^2}{b'^2}=1,\quad y^2=4a'x$$

is the same. In fact, the principal axes are particular cases of the axes in §§ 160, 168, 172, *e.g.*, the major and minor axes of an ellipse are two conjugate diameters at right angles.

## Exercise.

35. The tangents at points $P$, $P'$ on a parabola meet in $T$, $U$ is the mid-point of $PP'$, and $TU$ meets the parabola in $Q$, show that $TQ = QU$.

[Take for axes the diameter bisecting $PP'$ and the tangent at its extremity, the equation is $y^2 = 4a'x$, $Q$ being the origin, and $P$, $P'$ are $(x_1, y_1)$, $(x_1, -y_1)$.]

## Illustrative Examples.

(i.) *To find the locus of the middle points of focal chords of a parabola.*

Let $(x_1, y_1)$ be the middle point of a chord of $y^2 = 4ax$. Then, by § 152, the equation of the chord is

$$yy_1-2ax = y_1^2-2ax_1.$$

This must pass through the focus $(a, 0)$, and we must have

$$y_1^2-2ax_1 = -2a^2.$$

Therefore the equation of the locus is

$$y^2 = 2a(x-a),$$

so that the locus is a parabola having the same axis as the original one, but its vertex is at the original focus $(a, 0)$, and its latus rectum is $2a$.

(ii.) *To find the middle point of the portion of the line $y = mx + c$ intercepted by the conic $x^2/a^2 + y^2/b^2 = 1$.*

The following method is applicable in general:—

Let $(x_1, y_1)$, $(x_2, y_2)$ be the points of intersection of the line and the curve. To find the abscissæ we eliminate $y$ between the two equations above, and thus obtain the quadratic

$$\frac{(mx+c)^2}{b^2}+\frac{x^2}{a^2}=1$$

or

$$x^2\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right)+\frac{2mc}{b^2}x+\frac{c^2}{b^2}-1=0.$$

Hence

$$x_1+x_2=-\frac{2mc}{b^2}\Big/\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right);$$

but, if $\xi$, $\eta$ be the coordinates of the middle point, we have

$$\xi = \frac{x_1 + x_2}{2} = -\frac{mc}{b^2} \Big/ \left(\frac{1}{a^2} + \frac{m^2}{b^2}\right),$$

$$\eta = m\xi + c = -\frac{m^2 c}{b^2} \Big/ \left(\frac{1}{a^2} + \frac{m^2}{b^2}\right) + c.$$

Since the ratio $\eta/\xi$ does not depend on $c$, this gives another proof of the fact that the locus of the middle point of a system of parallel chords is a diameter.

(iii.) *A tangent to an ellipse meets the director circle in* **P**, **Q**. *Show that, if* **C** *be the centre, then* **CP**, **CQ** *are conjugate diameters of the conic.*

Suppose the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{..............................} \quad \text{(i.)}.$$

Then the equation of the director circle is

$$x^2 + y^2 = a^2 + b^2 \quad \text{..............................} \quad \text{(ii.)}.$$

The tangent at $(x_1, y_1)$ is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ .............................(iii.).

Hence, since $C$ is the origin, the equation of $CP$, $CQ$ is obtained by combining (ii.) and (iii.) so as to get an equation homogeneous in and $y$.

Therefore the equation is

$$x^2 + y^2 = (a^2 + b^2)\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2}\right)^2$$

or $x^2 \left\{ (a^2 + b^2)\dfrac{x_1^2}{a^4} - 1 \right\} + 2xy\dfrac{x_1 y_1}{a^2 b^2}(a^2 + b^2) + y^2 \left\{ (a^2 + b^2)\dfrac{y_1^2}{b^4} - 1 \right\} = 0.$

But $\qquad y - mx = 0, \quad y - m'x = 0$

are conjugate diameters if $\quad mm' = -b^2/a^2$

and $\qquad mm' = -\dfrac{\text{coeff. of } x^2}{\text{coeff. of } y^2}$

in the above equation.

Therefore the lines are conjugate diameters if

$$\frac{1}{a^2}\left\{ (a^2 + b^2)\frac{y_1^2}{b^4} - 1 \right\} + \frac{1}{b^2}\left\{ (a^2 + b^2)\frac{x_1^2}{a^4} - 1 \right\} = 0,$$

that is, if $\qquad \left(\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2}\right)(a^2 + b^2)\dfrac{1}{a^2 b^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$

which is satisfied, since $(x_1, y_1)$ is on the conic.

(iv.) *Two tangents to a parabola meet the tangent at a fixed point $O$ in points $Q$ and $R$ such that $OQ \,.\, OR$ is constant. Find the locus of their point of intersection.*

Take for axes the tangent at $O$ and the diameter through $O$, then the equation of the parabola is of the form

$$y^2 = 4ax.$$

Suppose $P$ is a point on the locus required; then

$$OQ \,.\, OR = \text{const.} = k^2, \text{ say.}$$

Let the coordinates of $P$ be $x_1, y_1$; then the tangents from it to the curve are

$$\{yy_1 - 2a(x + x_1)\}^2 = (y^2 - 4ax)(y_1^2 - 4ax_1).$$

To find where these tangents meet the fixed tangent we must put $x = 0$, and we find that $OQ$, $OR$ are the roots of the quadratic

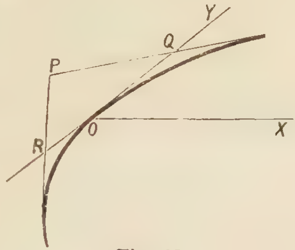



Fig. 62.

$$(yy_1 - 2ax_1)^2 = y^2(y_1^2 - 4ax_1)$$

or

$$y^2 \,.\, 4ax_1 - 4ax_1y_1 \,.\, y + 4a^2x_1^2 = 0;$$

$$\therefore \quad OQ \,.\, OR = \frac{4a^2x_1^2}{4ax_1} = ax_1.$$

But

$$OQ \,.\, OR = k^2.$$

Therefore the locus is given by $ax_1 = k^2$; or dropping the suffix it is $ax = k^2$, a line parallel to the fixed tangent.

## MISCELLANEOUS EXERCISES ON CHAP. XIII.

36. Obtain the locus of the middle point of parallel chords of the conic $x^2 + xy + y^2 + 2x = 0$ which are parallel to the line $x = y$.

37. Prove, in any way, that the locus of the middle points of chords of

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

which are parallel to the axis of $x$ is the straight line $ax + hy + g = 0$.

38. The equation of a hyperbola is $xy + 2gx + 2fy + c = 0$. Find the equations of its asymptotes and its conjugate hyperbola.

**39. If $SY$, $S'Y'$ be the perpendiculars from the foci on the tangent at a point $P$ to the ellipse $x^2/a^2 + y^2/b^2 = 1$, prove that**

$$SY \,.\, S'Y' = b^2, \quad SP \,.\, S'P = CD^2,$$

**where $CD$ is the semi-diameter conjugate to $CP$.**

40. A line is drawn through a point $O(x_1, y_1)$ to meet the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the points $P$ and $Q$. If $R$ is the middle point of $PQ$, and the chord makes an angle $\theta$ with $OX$, then show that

$$OR = -\left(\frac{x_1 \cos\theta}{a^2} + \frac{y_1 \sin\theta}{b^2}\right) \Big/ \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right).$$

Deduce the coordinates of $R$.

41. Find the condition necessary in order that the lines $Ax + By + C = 0$, $A'x + B'y + C' = 0$ may be parallel to a pair of conjugate diameters of the ellipse $4x^2 + 9y^2 = 36$.

42. The points on the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, the tangents at which are parallel to $y = mx$, are the extremities of the diameter $(ax + hy + g) + m(hx + by + f) = 0$.

43. The angle between the equal conjugate diameters in an ellipse is 60°. Prove that the eccentricity of the curve is $\frac{1}{3}\sqrt{6}$.

44. Find the locus of the point of intersection of the perpendiculars to a pair of conjugate diameters of an ellipse, one drawn from one focus, the other from the other focus.

45. Trace the curve $xy = bx + ay$, and find the locus of the middle points of the chords parallel to the line $y = x$.

46. Find the locus of the middle point of a focal chord of the parabola $y^2 = 4ax$.

47. Find the condition that $y = mx$, $y = m_1 x$ should be parallel to conjugate diameters of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

48. Find the equations of the straight lines conjugate to the $x$ and $y$ axes respectively in the curve $Ax^2 + 2Bxy + Cy^2 = 1$. Find the conditions that these lines should coincide, and interpret the result.

49. **Interpret the equations $x^2 \pm y^2 = a^2$ referred to oblique axes inclined at an angle $\omega$.** Find the lengths of the semi-axes.

50. The axes being oblique, show that the equation $y^2 = 2cx - kx^2$ represents an ellipse, a parabola, or a hyperbola according as $k$ is positive, zero, or negative. Also the axis of $x$ is a diameter, and the axis of $y$ is the tangent at its extremity.

51. The tangents from a variable point $P$ to a conic meets the tangent at a fixed point $A$ in points $Q$ and $R$ such that $AQ \,.\, AR$ is constant. Show that the locus of $P$ is a straight line parallel to the tangent at $A$.

52. $PCP'$, $DCD'$ are any pair of conjugate diameters of a given ellipse. If $Q$ be the middle point of $PD$, find the locus of $Q$; and show that the area of the parallelogram $PDP'D'$ is constant.

53. Prove that, if $P$ be any point on a central conic, $A$ and $A'$ the extremities of the major axis, then $PA$ and $PA'$ are parallel to conjugate diameters.

Does the same property hold for $PB$, $PB'$?

54. Chords drawn from any point $Q$ on an ellipse to the extremities of any diameter $PCP'$ intersect its conjugate $DCD'$ in $M$, $N$. Prove that $CM \,.\, CN = CD^2$.

55. A series of parallelograms is inscribed in an ellipse whose sides are parallel to the equal conjugate diameters. Prove that the sum of the squares on the sides is the same for all the parallelograms.

56. Show that, if the asymptotes of the first of two conics are parallel to conjugate diameters of the second, then the asymptotes of the second are parallel to conjugate diameters of the first.

57. Find the equation of the locus of the middle points of all chords of the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ which pass through the origin.

58. Prove that the straight line $y = mx$ bisects all chords of the hyperbola $xy = k^2$ which are parallel to $y = -mx$.

59. Prove that the pair of straight lines whose equation is

$$(h'a-ha')\,x^2+(ab'-a'b)\,xy+(hb'-h'b)\,y^2=0$$

will be conjugate diameters of $ax^2+2hxy+by^2=1$ and also of the conic $a'x^2+2h'xy+b'y^2=1$.

[See Ex. 13, § 157.]

60. If a fixed straight line, parallel to either axis of the ellipse, be met by a pair of conjugate diameters in the points $K$, $L$, show that the circle described on $KL$ as diameter passes through two fixed points on the other axis.

61. If a pair of conjugate semi-diameters of an ellipse intersect the tangents at the extremities of the major axis in the points $Q$, $R$, then $QR$ touches the ellipse.

62. Through a fixed point, any number of chords of a parabola are drawn. Show that their mid-points all lie on a certain parabola whose latus rectum is half as long as that of the given parabola.

63. Show that, if $MN$ is one of a system of parallel chords of an ellipse parallel to $y = mx$, and a point $P$ be taken on it such that $MP : PN = 1 : 3$, then the locus of $P$ is a concentric ellipse.

# CHAPTER XIV.

## NORMALS TO CONICS.

[Throughout this chapter the axes of coordinates will be supposed rectangular.]

174. **Normal.**—DEFINITION.—The **normal** at any point of a conic is the line drawn through that point perpendicular to the tangent there.

The same definition applies to any curve. As examples, we may note that the normal at any point of a straight line is perpendicular to it, and the normal at any point of a circle is the radius through that point.

**175. To find the equation of the normal at the point $(x_1, y_1)$ on the conic**

$$\boldsymbol{ax^2+2hxy+by^2+2gx+2fy+c=0.}$$

The tangent at the point $(x_1, y_1)$ is

$$x\,(ax_1+hy_1+g)+y\,(hx_1+by_1+f)+gx_1+fy_1+c=0. \quad (\S\ 127)$$

Now the line whose equation is

$$\frac{x}{ax_1+hy_1+g}-\frac{y}{hx_1+by_1+f}=k$$

is perpendicular to the tangent for all values of $k$ (Part I., § 19).

If it passes through the point $(x_1, y_1)$, we must have

$$k=\frac{x_1}{ax_1+hy_1+g}-\frac{y_1}{hx_1+by_1+f}.$$

Consequently the required equation is

$$\boldsymbol{\frac{x-x_1}{ax_1+hy_1+g}=\frac{y-y_1}{hx_1+by_1+f}} \quad \ldots\ldots\ldots\ldots (1),$$

which might have been written down at once.

**176. Particular cases.**

The reader should now obtain the following results from first principles :—

(i.) Parabola, $y^2 = 4ax$;
normal, $2a(y-y_1)+y_1(x-x_1) = 0$.

(ii.) Ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; normal $\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$.

(iii.) Hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; normal $\frac{x-x_1}{x_1/a^2}=-\frac{y-y_1}{y_1/b^2}$.

**177.** The equation of the normal to the ellipse

$$x^2/a^2+y^2/b^2 = 1$$

is $$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2} \quad \text{or} \quad \frac{a^2x}{x_1}-a^2=\frac{b^2y}{y_1}-b^2,$$

*i.e.* $$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2 \quad\text{..................}\quad (2),$$

in which form it should be remembered.

Similarly, the normal to the hyperbola is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2 \quad\text{..................}\quad (3).$$

*Example.—The point* (1, 2) *is on the conic*

$$x^2+xy+y^2+x+y = 10.$$

*To find the equation of the normal at that point.*

The general equation of the tangent at $(x_1, y_1)$ is

$$x(ax_1+hy_1+g)+y(hx_1+by_1+f)+(gx_1+fy_1+c) = 0,$$

which here becomes

$$x(x_1+\tfrac{1}{2}y_1+\tfrac{1}{2})+y(\tfrac{1}{2}x_1+y_1+\tfrac{1}{2})+\tfrac{1}{2}x_1+\tfrac{1}{2}y_1-10 = 0$$

or $$x(2x_1+y_1+1)+y(x_1+2y_1+1)+x_1+y_1-20 = 0,$$

*i.e.* the tangent at (1, 2) is $5x+6y-17 = 0$;
therefore the normal is $6(x-1)-5(y-2) = 0$
or $$6x-5y+4 = 0.$$

To test the accuracy of his work, the reader should always verify that the normal actually passes through the given point.

**Exercises.**

1. Find the equation of the normal at the point (1, 1) on the conic
$$x^2+2y^2+x+y = 5.$$

2. Find the equation of the normal at the point $(\frac{1}{2}, \frac{1}{3})$ on the conic
$$2x^2+6xy+9y^2+x+y = \tfrac{10}{3}.$$

3. Show that the conic
$$ax^2+2hxy+by^2+2gx+2fy = 0$$
passes through the origin, and the normal there is $fy-gx = 0$.

4. Write down the equation of the normal at any point of the conic
$$ax^2+2hxy+by^2 = 1.$$
Find the condition that the normal should pass through the origin, and interpret the result geometrically.

5. **If the normal at $P$ to an ellipse meet $SS'$ in $G$, and $PN$ be perpendicular to $SS'$, then show that**
$$CG = e^2 . CN.$$

6. **The normal at $P$ bisects the internal angle between the focal distances of $P$.**

7. **Show that $CG = e^2 . CN$ is true for the hyperbola, but that the normal now bisects the external angle between the focal distances.**

8. **If the normal at $P$ to a parabola meet the axis in $G$, and $PN$ be the ordinate, then the *subnormal* $NG$ is constant.**

9. **If a tangent and normal be drawn at any point on a parabola, prove that they meet the axis in two points which are equidistant from the focus.**

10. In the parabola $y^2 = 4ax$, the normal at $(x_1, y_1)$ meets the curve again in the point whose coordinates are
$$\frac{(y_1{}^2+8a^2)^2}{4ay_1{}^2}, \quad -\left(\frac{y_1{}^2+8a^2}{y_1}\right).$$
[Substitute $y^2/4a$ and $y'^2/4a$ for $x$ and $x'$ in equation (1).]

11. If the normal at $P(x', y')$ to the parabola $y^2 = 4ax$ meet the curve again at $Q$, find the length of $PQ$.

12. Show that, if the normal to an ellipse at the end of the latus rectum passes through one end of the minor axis, then
$$a^2+ab-b^2 = 0.$$
Hence deduce that the eccentricity is given by
$$e^4+e^2-1 = 0.$$

178. **If the normal at $(x_1, y_1)$ on the ellipse $x^2/a^2+y^2/b^2 = 1$ makes an acute angle $\theta$ with the axis of $x$, and $p$ be the perpendicular from the centre on the tangent at that point, then** $\cos\theta = \dfrac{px_1}{a^2}$, $\sin\theta = \dfrac{py_1}{b^2}$.

Since the equation of the normal is

$$\frac{x-x_1}{x_1/a^2} = \frac{y-y_1}{y_1/b^2},$$

comparing this with the form

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} \qquad \text{(Part I., § 10, B)}$$

we have

$$\frac{\cos\theta}{x_1/a^2} = \frac{\sin\theta}{y_1/b^2}.$$

Hence

$$\cos\theta = \frac{x_1/a^2}{\sqrt{x_1^2/a^4+y_1^2/b^4}}$$

and

$$\sin\theta = \frac{y_1/b^2}{\sqrt{x_1^2/a^4+y_1^2/b^4}}.$$

But the tangent at $(x_1, y_1)$ is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$,

and hence

$$p = \pm\frac{1}{\sqrt{x_1^2/a^4+y_1^2/b^4}}.$$

Hence, as $\theta$ is acute, we have

$$\cos\theta = \frac{px_1}{a^2}, \quad \sin\theta = \frac{py_1}{b^2} \quad \text{.................. } (\mathbf{4}).$$

Similarly, for the hyperbola $x^2/a^2-y^2/b^2 = 1$, we have

$$\cos\theta = \frac{px_1}{a^2}, \quad \sin\theta = -\frac{py_1}{b^2},$$

as the reader can easily verify.

179. Various results can be deduced from the foregoing formulæ for $\cos\theta$ and $\sin\theta$.

**I. If the normal at $P$ meet the axes in $G$ and $F$, then**

$$PG = b^2/p, \quad PF = a^2/p.$$

For the coordinates of a point on the normal at a distance $r$ from $P$, measured inwards, are

$$x_1 - r\cos\theta, \quad y_1 - r\sin\theta,$$

and these, by § 178, are equal to

$$x_1 - r\,\frac{px_1}{a^2}, \quad y_1 - r\,\frac{py_1}{b^2}.$$

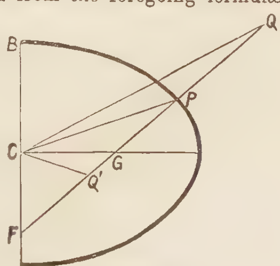


Fig. 63.

If $r = PG$, the second of these is zero, and therefore

$$\frac{rp}{b^2} = 1 \quad \text{or} \quad r = \frac{b^2}{p}.$$

If $r = PF$, the first is zero, and therefore

$$\frac{rp}{a^2} = 1 \quad \text{or} \quad r = \frac{a^2}{p}.$$

**II. If we measure lengths $PQ$, $PQ'$ outwards and inwards each equal to $CD$, the semi-diameter conjugate to $CP$, then**

$$CQ = a+b, \quad CQ' = a-b,$$

**and $CQ$, $CQ'$ are equally inclined to the axes.**

In fact, suppose that $Q$ is $(\xi, \eta)$ and $Q'$ is $(\xi', \eta')$ ; then, if $d = CD$,

$$\xi = x' + d\frac{px'}{a^2}, \quad \eta = y' + d\frac{py'}{b^2};$$

$$\xi' = x' - d\frac{px'}{a^2}, \quad \eta' = y' - d\frac{py'}{b^2};$$

but $$pd = ab; \qquad (\S\ 158)$$

$$\therefore \quad \xi = x'\left(1 + \frac{b}{a}\right) = \frac{x'}{a}(a+b);$$

$$\eta = y'\left(1 + \frac{a}{b}\right) = \frac{y'}{b}(a+b).$$

Similarly,

$$\xi' = \frac{x'}{a}(a-b),$$

and $$\eta' = -\frac{y'}{b}(a-b).$$

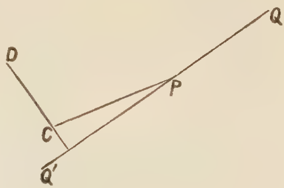


Fig. 64.

Consequently

$$CQ^2 = \xi^2 + \eta^2$$

$$= (a+b)^2\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) = (a+b)^2;$$

$$\therefore \quad CQ = a+b. \quad \text{Similarly,} \quad CQ = a-b;$$

and, since $$\frac{\eta}{\xi} = -\frac{\eta'}{\xi'},$$

the lines $CQ$, $CQ'$ are equally inclined to the axes.

**III. To construct the axes of an ellipse, having given two conjugate diameters in magnitude and position.**

Let $CP$, $CD$ be the given diameters ; then, since the normal at $P$ is perpendicular to $CD$ (§ 157), we can easily find $Q$ and $Q'$, for

$$PQ = PQ' = CD.$$

Then, by II., the axes bisect the internal and external angles between $CQ$ and $CQ'$, and

$$2a = CQ + CQ', \quad 2b = CQ - CQ'$$

Exactly similar results may be established for the hyperbola by making use of the corresponding formulæ for $\cos\theta$ and $\sin\theta$. These we leave as an exercise for the reader.

**180. To show that three normals can be drawn from a given point to a parabola.**

The equation of the normal at $(x', y')$ is

$$2ay + xy_1 = x_1y_1 + 2ay_1,$$

or, since $x_1 = \dfrac{y_1^2}{4a}$, this is

$$2ay + xy_1 = \frac{y_1^3}{4a} + 2ay_1.$$

Now if the normal passes through the point $(f, g)$, we have

$$2ag + fy_1 = \frac{y_1^3}{4a} + 2ay_1$$

or

$$\frac{y_1^3}{4a} + y_1(2a - f) - 2ag = 0,$$

an equation of the third degree for $y$. Hence, as there are three roots (*Tut. Alg.*, II., § 371), and for each root a point on the curve, there are three points such that the normals at them pass through a given point.

Cor.—**If the normals at three points are concurrent, the sum of their ordinates is zero.**

For in the above equation there is no term in $y_1^2$, and hence the sum of the roots is zero (*Tut. Alg.*, II., § 374).

### Exercises.

13. The normal at the point (1, 1) in the ellipse $x^2 + 3y^2 = 4$ makes an angle $\alpha$ with the major axis. Find the values of $\cos\alpha$, $\sin\alpha$.

14. Find the cosine and the sine of the angle that the normal at $(1, \frac{1}{2})$ on $x^2 + 4y^2 = 2$ makes with the major axis.

15. A chord of a rectangular hyperbola which subtends a right angle at a given point is parallel to the normal at that point.

16. Find the coordinates of the points on the parabola $y^2 = 4x$ the normals at which pass through the point $(\frac{15}{4}, -\frac{3}{4})$, and draw a figure showing three concurrent normals.

[Here obtain the cubic for $y$, and to solve notice that one root is unity, so the others are given by a quadratic.]

17. Determine the coordinates of the points on the ellipse $x^2/a^2 + y^2/b^2 = 1$ at which the normal makes an angle of 45° with the axis of $x$.

18. Prove that at least one real normal can be drawn from any point to a parabola.

[An equation of the third degree has at least one real root.]

**181. Equation to the normal to a parabola in the form** $y = mx + c$.

We shall now find the equation of the normal to a parabola $y^2 = 4ax$ in terms of the tangent of the angle the normal makes with the axis of $x$.

Its equation is of the form

$$y = mx + c.$$

If it is the normal at $(x', y')$, we have

$$y' = mx' + c,$$

and it is perpendicular to the tangent at $(x', y')$, *i.e.* to

$$yy' = 2a(x + x');$$

$$\therefore \quad m = -y'/2a \quad \text{or} \quad y' = -2am;$$

and $$x' = y'^2/4a = am^2;$$

$$\therefore \quad c = y' - mx' = -2am - am^3.$$

Therefore the equation of the normal is

$$\boldsymbol{y = mx - 2am - am^3} \quad \ldots\ldots\ldots\ldots\ldots \textbf{(5)}.$$

**Exercises.**

19. Show that the normal at any point of a parabola (terminated by the axis) is equal to the ordinate through the middle point of the subnormal.

20. Find the equations of the normals to $y^2 = 4ax$ that make angles (i.) 30°, (ii.) 45°, (iii.) 120°, with the axis.

21. Find the equation of the normal to $y^2 = 4a(x-a)$ that makes an angle of 45° with the axis.

[Transfer origin to $(a, 0)$, and finally transfer back again.]

22. Deduce the results of § 180 from the normal equation

$$y = mx - 2am - am^3.$$

**182. Equation of a conic referred to a tangent and the corresponding normal as axes of coordinates.**

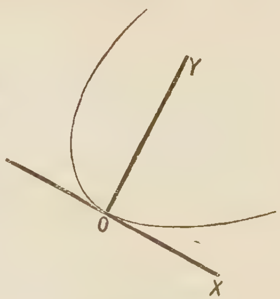


Fig. 65.

Let the tangent be the axis of $x$ and the normal the axis of $y$.

Then the equation of the conic is of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0;$$

but, since the axis of $x$ ($y=0$) meets the curve in two points coinciding with the origin, the equation

$$ax^2+2gx+c=0$$

must have both roots zero; hence

$$g=0 \quad \text{and} \quad c=0.$$

Consequently the required equation is

$$ax^2+2hxy+by^2+2fy=0 \quad \ldots\ldots\ldots\ldots \quad \textbf{(6)},$$

a form which is often useful, one advantage being that the axes are rectangular.

Many properties of conics can be deduced most easily by using these axes, as, for instance, in the following example:—

### Illustrative Examples.

(i.) **Chords of a conic which subtend a right angle at a fixed point of the curve pass through a fixed point on the normal at that point.**

This is clearly a case in which it is advantageous to use the tangent and normal at the point as axes.

Suppose the conic is $\quad ax^2+2hxy+by^2+2fy=0,$

and that a chord is $\quad lx+my=1.$

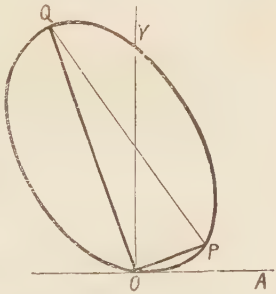


Fig. 66.

The equation of the lines joining the extremities $P$, $Q$ of this chord to the origin $O$ is (by Part I., § 38)

$$ax^2+2hxy+by^2+2fy\,(lx+my)=0$$

or $\quad ax^2+2xy\,(h+fl)+by^2\,(b+2fm)=0.$

But, since this equation represents two lines at right angles, we have

$$a+b+2fm=0 \quad (\text{Pt.I.}, §29)$$

or $$m=-\frac{a+b}{2f}.$$

Therefore $m$ is the same for all such chords.

But $lx+my=1$ meets the normal ($x=0$) at a point whose ordinate is given by $my=1$ or $y=\dfrac{1}{m}$, which is constant.

Therefore all such chords pass through a fixed point on the normal.

(ii.) *Find the locus of the point of intersection of two normals to the parabola* $y^2 = 4ax$ *that are perpendicular to each other.*

Taking the equation $y = mx - 2am - am^3$,

if the normal passes through $(h, k)$, we have

$$k = mh - 2am - am^3.$$

This gives *three* values for $m$ (say $m_1$, $m_2$, $m_3$), and by the theory of equations, writing the equation in the form

$$m^3 + \frac{2a-h}{a}m + \frac{k}{a} = 0,$$

we have

$$m_1 + m_2 + m_3 = 0 \text{ .............................. } (1),$$

$$m_1m_2 + m_1m_3 + m_2m_3 = \frac{2a-h}{a} \text{ ......................... } (2),$$

$$m_1m_2m_3 = -\frac{k}{a} \text{ ......................... } (3).$$

But, by hypothesis, two of the normals are at right angles, and therefore

$$m_1m_2 = -1.$$

Therefore, by (3),

$$m_3 = \frac{k}{a};$$

and, substituting in (1) and (2), we have

$$m_1 + m_2 = -\frac{k}{a},$$

$$-1 + (m_1 + m_2)\frac{k}{a} = \frac{2a-h}{a}.$$

Eliminating $m_1 + m_2$, we get

$$-1 - \frac{k^2}{a^2} = \frac{2a-h}{a} \quad \text{or} \quad k^2 + 3a^2 - ah = 0.$$

Therefore locus of $(h, k)$ is

$$y^2 = ax - 3a^2.$$

## MISCELLANEOUS EXERCISES ON CHAP. XIV.

23. Find the condition that the line $x \cos \alpha + y \sin \alpha - p = 0$ should be a normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$.

24. Find the points on the ellipse $x^2/a^2 + y^2/b^2 = 1$, the normals at which pass through a given point $(g, 0)$ on the major axis.

25. A distance $r$ is measured inwards along the normal to the ellipse $x^2/a^2+y^2/b^2 = 1$ at any point $P$, so that $pr = m^2$, where $p$ is the length of the perpendicular from the centre on to the tangent at the point. Find the coordinates of the point so obtained in terms of those of $P$.

Show also that the locus of the point is the ellipse

$$\frac{a^2x^2}{(a^2-m^2)^2}+\frac{b^2y^2}{(b^2-m^2)^2}=1.$$

26. Find the lengths of the normals to $y^2 = 4ax$ drawn from the point on the axis distant $8a$ from the focus.

27. Find the equation of the locus of the intersection of the normal to the parabola $y^2 = 4ax$ at any point $P$ on it with a straight line drawn parallel to the axis at the same distance from the axis as $P$, but on the opposite side.

28. The normal at $(x_1, y_1)$ to the general conic passes through the point whose coordinates are

$$x_1-\frac{2}{a+b}(ax_1+hy_1+g), \quad y_1-\frac{2}{a+b}(hx_1+by_1+f).$$

29. Show that the line $lx+my = 1$ will be a normal to the ellipse $x^2/a^2+y^2/b^2 = 1$ if $\quad \dfrac{a^2}{l^2}+\dfrac{b^2}{m^2}=(a^2-b^2)^2.$

What does this give for the circle?

30. Find the condition for $lx+my = 1$ to be normal to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

31. Find the locus of the point of intersection of the perpendicular from the focus of a parabola to a normal.

32. If the normals at two points of the parabola $y^2 = 4ax$ make angles $\theta$, $\phi$ with the axis, such that $\tan\theta\tan\phi = 2$, show that the normals intersect on the parabola.

33. Find the locus of the middle point of the normal $PG$ to the parabola $y^2 = 4ax$.

34. If $X$, $Y$ be the coordinates of the middle point of the intercept of the normal to $x^2/a^2+y^2/b^2 = 1$ made by the coordinate axes, prove that $\quad a^2X^2+b^2Y^2 = \frac{1}{4}(a^2-b^2)^2.$

35. Prove that the distance between a tangent to the parabola and the parallel normal is $a\operatorname{cosec}\theta\sec^2\theta$, where $\theta$ is the angle which either makes with the axis.

36. $P$ is any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. $PQ$ is drawn parallel to the axis of $x$, cutting the ellipse again at $Q$. $PR$ is drawn parallel to the axis of $y$, cutting the ellipse again at $R$. Prove that the locus of the intersection of the straight line $QR$ with the normal at $P$ is the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$

37. If $OP$, $OP'$ be two chords of a conic inclined at angles $\theta$, $\theta'$ to the normal at a given point $O$, prove that, if $\tan\theta \,.\, \tan\theta'$ be constant, the chord $PP'$ will intersect the normal at a constant point.

38. In the rectangular hyperbola the portion of a normal intercepted between the axes is bisected by the curve.

39. A chord of a conic moves so that the lines $OP$, $OQ$ joining its extremities to a fixed point of the conic are equally inclined to the normal at that point. Show that the chord meets the tangent at $O$ in a fixed point.

40. If $(x_1, y_1)$ be on the ellipse $x^2/a^2+y^2/b^2=1$, show that
$$\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}\right)-\frac{1}{a^2b^2}.$$

Hence show that, if the normal at $P$ $(x_1, y_1)$ meet the ellipse again in $Q$, we have
$$pr\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}\right)=2\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}\right),$$
where $p$ is the central perpendicular on the tangent at $P$ and $r=PQ$, so that
$$r=\frac{2a^2b^2}{p\,(a^2+b^2-p^2)}.$$

[The coordinates of $Q$ are $x_1-\frac{px_1}{a^2}r$, $y_1-\frac{py_1}{b^2}r$. Put down the condition that $Q$ is on the curve.]

41. Show that the locus of the middle point of the normal $PG$ to the ellipse $x^2/a^2+y^2/b^2=1$ is
$$x^2/a^2+(1+e^2)^2y^2/b^2=(1+e^2)^2/4.$$

42. Show that, if the normal to a rectangular hyperbola at a variable point $P$ meet the curve again in $Q$, $PQ \propto CP^3$, where $C$ is the centre.

43. The three normals to a parabola $y^2=4ax$ drawn from the point $(h, k)$ meet the axis in $G_1$, $G_2$, and $G_3$ respectively. Show that $AG_1+AG_2+AG_3=2\,(h+a)$, where $A$ is the vertex of the curve.

44. Normals are drawn to a parabola $y^2=4ax$ from a given point $Q$, inclined at angles $\theta_1$, $\theta_2$, $\theta_3$ to the axis. Show that
$$SQ=a\sec\theta_1\sec\theta_2\sec\theta_3.$$

## EXAMINATION PAPER IV.

1. (*a*) Find the equation of the chord joining the points $(p, q)$, $(p', q')$ on the curve
$$mxy + by^2 - 2gx + 2fy - c = 0,$$
and hence deduce the equation of the tangent at the point $(p, q)$.

(*b*) Find the equation of the tangents drawn from the point $(x_1, y_1)$ to the curve $x^2/a^2 + y^2/b^2 = 1$. Hence deduce the equation of the director circle.

2. Prove that the part of the tangent to a hyperbola intercepted between the asymptotes is bisected by the point of contact, and that the area cut off is constant.

3. Find the equation of the tangent to the parabola $y^2 = 4ax$ at a given point $(h, k)$, and show that one and only one tangent can be drawn having a given inclination to the axis of symmetry.

Show that tangents to a given parabola which are inclined to each other at an angle of 45° intersect on a rectangular hyperbola.

4. Find, from first principles, the locus of the middle points of chords of $3x^2 + 2xy + 4x + 7y + 1 = 0$ parallel to $y = 3x$.

5. Define *conjugate diameter*, *conjugate hyperbola*; and prove that the tangents at the ends of a diameter of a central conic are parallel to the conjugate diameter.

Find the equation of the hyperbola conjugate to
$$3x^2 - 4xy + 2x - 3y + 7 = 0.$$

6. If tangents be drawn to a hyperbola and its conjugate from a point on either asymptote, the points of contact will lie at the end of conjugate diameters.

7. Prove that the difference of the squares on a diameter of a hyperbola and its conjugate diameter is constant.

8. Find the equation of a parabola referred to a diameter and the tangent at its extremity as axes.

9. What form does the equation of a curve of the second degree take when the curve is referred to a tangent and normal as axes?

10. Show that normals at the ends of a focal chord of an ellipse meet on the straight line through the middle point of the chord parallel to the axis.

# CHAPTER XV.

## POLES AND POLARS.

183. **To show that two tangents can be drawn from any point to a conic, and to find their points of contact.**

The first part of this proposition has already been proved, for we have found the equation of the two tangents. (§ 138.)

We shall now give another investigation which leads to an important result.

Let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

and let $x_1, y_1$ be the coordinates of the point from which the tangents are to be drawn.

Suppose $(m, n)$ is the point of contact of one such tangent; then the equation of the tangent is (§ 127)

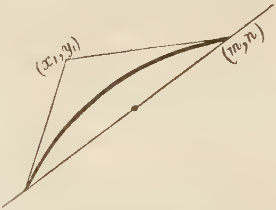


Fig. 65.

$$x(am + hn + g) + y(hm + bn + f) + gm + fn + c = 0.$$

Now, since this tangent passes through the point $(x_1, y_1)$,

$$x_1(am + hn + g) + y_1(hm + bn + f) + gm + fn + c = 0,$$

or, on rearranging,

$$m(ax_1 + hy_1 + g) + n(hx_1 + by_1 + f) + gx_1 + fy_1 + c = 0 \,.\, (\text{A}).$$

This is one equation for the two unknown quantities $m$ and $n$, and a further equation is, of course,

$$am^2 + 2hmn + bn^2 + 2gm + 2fn + c = 0 \ldots\ldots\ldots (\text{B}).$$

Thus, to find the points of contact, we have to solve the two equations (A) and (B) for $m$ and $n$. Since (A) is of

the first, and (B) of the second, degree in the unknowns, we can eliminate $m$ from (B) by means of (A) and obtain a quadratic for $n$. We have, therefore, two solutions, which we can easily find if we want them, and hence there are two tangents.

We regard their points of contact as found when we have sufficient equations to determine their coordinates.

**184. Geometrical interpretation of the equations.**

The two equations (A) and (B) both admit of very simple geometrical interpretations.

(B), of course, means that the point $(m, n)$ is on the conic; while (A) clearly means that $(m, n)$, the point of contact, must be on the straight line whose equation is

$$x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) + gx_1 + fy_1 + c = 0.$$

We infer at once that the points of contact are given by the points of intersection of this line and the conic.

**Hence the equation of the line joining the points of contact of the tangents from $(x_1, y_1)$ to the conic is**

$$x(ax_1 + hy_1 + g) + y(hx_1 + by_1 + f) + gx_1 + fy_1 + c = 0 \qquad \text{......... (1).}$$

[This holds whether the axes are rectangular or oblique.]

185. **Polar.** — DEFINITION. — The line joining the points of contact of the tangents drawn from a given point to a conic is called the **polar** of that point with respect to the conic. In Fig. 68 *PQ* is the polar of *T*.

**Pole.**—The point of intersection of the tangents at the extremities of a chord of a conic is called the **pole** of the chord. In Fig. 68 *T* is the pole of *PQ*.


Fig. 68.

Thus, if a line be the polar of a point, the point is the pole of the line.

NOTE.—It must be carefully noticed that **the equation of the polar is of exactly the same *form* as that of the tangent,** and thus need not be remembered separately. *The important difference between the two lines is that, whereas in the case of the tangent $(x_1, y_1)$ was on the curve, no such restriction is now imposed.*

It follows, of course, that

*when a point is* ON *the curve, its polar is the same as its tangent.*

This is also clear from geometrical considerations, because, as the point $T$ gradually approaches the curve, the points of contact become closer and closer together, and the line joining them ultimately becomes a tangent to the curve when $T$ is on the curve. Thus the tangent is only a particular case of the polar.

186. There is another point in the above connection to which the attention of the reader must be directed. The polar of a point has been defined as the line joining the points of contact of the tangents drawn from a point to a conic. Now if, for example, the conic be an ellipse and the point be inside it, the tangents are imaginary. It appears, however, from the equation of the polar that the latter is still a real line. The explanation is that, *although it is a real line, it does not cut the curve in real points, and hence the points of contact are imaginary though the line joining them is real.*

An example may make this clearer. The point (3, 3) is inside the ellipse $x^2+2y^2 = 36$, and the tangents drawn from it to the curve are accordingly imaginary. Let us actually find the points of contact and deduce the equation of the line joining them.

Suppose $(x_1, y_1)$ is a point of contact; then the tangent is

$$xx_1+2yy_1 = 36,$$

and, accordingly, $3x_1+6y_1 = 36$ or $x_1+2y_1 = 12$,

so that we have to solve the two equations

$$x_1+2y_1 = 12, \quad x_1^2+2y_1^2 = 36.$$

Eliminating $x_1$ and solving the quadratic for $y_1$, we find

$$y_1 = 4 \pm \sqrt{-2};$$

and hence, from the first equation,

$$x_1 = 4 \mp 2\sqrt{-2}.$$

Hence the points of contact are

$$(4+2\sqrt{-2},\ 4-\sqrt{-2}),\quad (4-2\sqrt{-2},\ 4+\sqrt{-2}),$$

and are imaginary, as we predicted. The line joining them is

$$\frac{x-4-2\sqrt{-2}}{4\sqrt{-2}}=\frac{y-4+\sqrt{-2}}{-2\sqrt{-2}},$$

which easily reduces to $x+2y-12=0$, a real line.

187. Similar remarks to those given in the last section apply to the pole of a given line. We have defined it as the point of intersection of the tangents at the points in which the line meets the conic. If the line does not meet the conic in real points, the tangents are imaginary; but, as we shall see, they intersect in a real point and the line has still a real pole. Thus the line $x+2y=12$ does not meet the ellipse $x^2+2y^2=36$ in real points, but its pole is nevertheless the real point (3, 3), as we have just seen.

188. **Equation of the polar in the simple cases.**

As particular cases of the general result, we can write down the equations of the polar in each of the simple cases, as follows:—

Parabola, $y^2-4ax=0$; polar, $\boldsymbol{yy_1-2a(x+x_1)=0}$.(**2**).

Ellipse, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$; polar, $\boldsymbol{\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1}$......(**3**).

Hyperbola, $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$; polar, $\boldsymbol{\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}=1}$......(**4**).

Hyperbola, $xy=c^2$; polar, $\boldsymbol{xy_1+x_1y=2c^2}$ ...(**5**).

In all cases the equation of the polar holds for oblique as well as for rectangular axes, for in obtaining it we have only used the equation of the tangent, and this holds even for oblique axes.

We shall now proceed to find, from first principles, the polar of $(x_1, y_1)$ to the ellipse $x^2/a^2+y^2/b^2=1$.

[**Caution.**—With regard to $xy=c^2$, it should be noted that, although the equation of the tangent at $(x_1, y_1)$ can be expressed in either of the forms $xy_1+x_1y=2c^2$, $x/x_1+y/y_1=2$, the latter equation does not in general give the equation of the polar.]

**189. To find the polar of a point with respect to $x^2/a^2+y^2/b^2=1$.**

Let the point be $O$ $(x_1, y_1)$, and let the tangents from $O$ to the ellipse be $OP$, $OQ$, and their points of contact $(m, n)$, $(m', n')$.

Consider the equation

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \qquad\text{(C)}.$$

This is an equation of the first degree, and therefore represents *some* straight line.

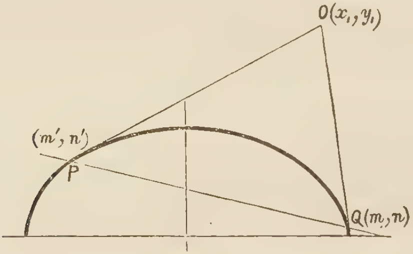



Fig. 69.

Now the equations to the tangents at $P$ and $Q$ are

$$\frac{xm}{a^2}+\frac{yn}{b^2}=1,\quad \frac{xm'}{a^2}+\frac{yn'}{b^2}=1;$$

and, since these tangents pass through $(x_1, y_1)$, we have

$$\frac{x_1m}{a^2}+\frac{y_1n}{b^2}=1,\quad \frac{x_1m'}{a^2}+\frac{y_1n'}{b^2}=1 \qquad\text{(D)}.$$

From (D) we see that the equation (C) is satisfied by the values $x=m$, $y=n$, and $x=m'$, $y=n'$, respectively.

Therefore
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

represents the straight line passing through $(m, n)$, $(m', n')$, *i.e.* it represents the straight line $PQ$.

In other words, the equation of the polar of $O$ $(x_1, y_1)$ is

$$\boldsymbol{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1} \qquad \textbf{(3)}.$$

190. For the parabola $y^2=4ax$ we may follow the general method of §§ 183, 184, or the slightly different one of § 189. Let us use the latter. Let the point $O$ be $(x_1, y_1)$, and let the tangents from $O$ to the parabola be $OP$, $OQ$, and their points of contact $(m, n)$, $(m', n')$. Consider the equation $$yy_1 = 2a(x+x_1) \quad\text{.....................}\quad (c).$$
This is an equation of the first degree, and therefore represents some straight line. Now the equation of the tangent at $(m, n)$ is $yn = 2a(x+m)$,
and this passes through $(x_1, y_1)$. Consequently
$$y_1 n = 2a(x_1+m),$$
which shows that $(m, n)$ is on the line (c). Similarly, $(m', n')$ is on this line. Thus the equation (c) represents the line joining the points of contact.

191. **The polar of any point with respect to a conic is parallel to the chord of the conic which is bisected at that point.**

By § 184, the polar of the point $(x_1, y_1)$ is
$$x(ax_1+hy_1+g)+y(hx_1+by_1+f)+gx_1+fy_1+c=0;$$
and, by § 153, the chord bisected at $(x_1, y_1)$ is
$$(x-x_1)(ax_1+hy_1+g)+(y-y_1)(hx_1+by_1+f)=0.$$
The coefficients of $x$ and $y$ in the two equations are identical, and therefore the lines are parallel. Of course, if the point be outside the conic, the chord bisected there meets the conic in imaginary points.

The result of this article may be easily established for the ellipse by referring the curve to a pair of conjugate diameters one of which passes through the point. For, by § 160, the equation of the ellipse will be
$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1,$$
and the point is $(c, 0)$. The chord bisected there is, by hypothesis, parallel to the conjugate diameter $x=0$. Its polar is $xc/a'^2=1$, and is therefore also parallel to $x=0$. Hence the two lines are parallel.

So, for the parabola, we can make use of § 172 in a similar way.

**Exercises.**

1. Find the equation of the polar of the point (1, 1) with respect to the conic $$x^2+2xy+2y^2-2x-4y+1=0.$$

2. Find the equation of the polar of the point $(\frac{1}{2}, \frac{1}{3})$ with respect to the conic $$x^2+xy+y^2+x+y+1=0.$$

3. Find the equation of the polar of the origin to the curve
$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

4. Following the method of §§ 183, 184, find from first principles the equations of the polars in the cases given in § 188.

5. **Show that the equation of a line through the point $P$ $(x_1, y_1)$, perpendicular to its polar with respect to $x^2/a^2+y^2/b^2=1$ is** $$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2.$$

What does this become when $(x_1, y_1)$ is on the conic?

6. The perpendicular in Ex. 5 meets the major axis in $G$, and $PN$ is perpendicular to the major axis. Show that
$$CG=e^2.CN.$$

7. Write down the equation of the polar of the point $P$ $(x_1, y_1)$ with respect to the parabola $y^2=4a(x+a)$.

If the polar meet the directrix in $T$, find the equation of the line joining $T$ to the origin. Hence (the focus being the origin) show that $PST$ is a right angle.

[Note that the equation of the directrix is $x+2a=0$.]

8. Verify that, although tangents from the point (2, 1) to the conic $x^2-y^2=2$ are imaginary, yet their chord of contact is the real line
$$2x-y=2.$$

9. Prove that the tangents from the point $(x_1, y_1)$ to the hyperbola $x^2/a^2-y^2/b^2=1$ are only real and distinct when $x^2/a^2-y^2/b^2<1$, but that the chord of contact is the straight line $xx_1/a^2-yy_1/b^2=1$, which is real in all cases.

We shall now proceed to give some of the **reciprocal properties of poles and polars.**

**192. If the polar of $A$ passes through $B$, then will the polar of $B$ pass through $A$.**

Let $A$ be $(x_1, y_1)$, $B$ $(x_2, y_2)$, and the equation of the conic
$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$
The equation of the polar of $A$ is

$$x(ax_1+hy_1+g)+y(hx_1+by_1+f)+gx_1+fy_1+c=0;$$

and, since this passes through $B$, we have

$$x_2(ax_1+hy_1+g)+y_2(hx_1+by_1+f)$$
$$+gx_1+fy_1+c=0$$

or $ax_1x_2+h(x_1y_2+x_2y_1)$
$$+by_1y_2+g(x_1+x_2)$$
$$+f(y_1+y_2)+c=0\ldots(\kappa),$$

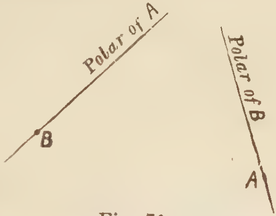



Fig. 70.

an equation which is perfectly symmetrical in the two sets of coordinates, and hence it is also the condition that the polar of $B$ should pass through $A$.

If necessary, we can give a formal proof of the latter portion. The equation ($\kappa$) may be written

$$x_1(ax_2+hy_2+g)+y_1(hx_2+by_2+f)+gx_2+fy_2+c=0,$$

and, as the equation of the polar of $B$ is

$$x(ax_2+hy_2+g)+y(hx_2+by_2+f)+gx_2+fy_2+c=0,$$

this shows that it passes through the point $A$ $(x_1, y_1)$, as was to be proved.

This theorem is easier to see if we take the conic in one of the simple forms, as, for example,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

For now the polar of $(x_1, y_1)$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,$$

and, if this passes through $(x_2, y_2)$, we have

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1,$$

a relation which is obviously symmetrical, and hence the polar of $(x_2, y_2)$ passes through $(x_1, y_1)$.

**Exercise.**

10. Prove, in like manner, the above theorem in each of the other simple cases (§ 188).

193. **Conjugate points.** — DEFINITION. — Two **points** are said to be **conjugate** with respect to a conic when the polar of each passes through the other.

**Conjugate lines.** — In like manner, two **lines** are **conjugate** when the pole of each lies on the other.

This last statement involves a theorem to be proved, namely that,

194. **If the pole $P$ of a line $p$ lies on a line $q$, then the pole $Q$ of the line $q$ lies on $p$.**

For the polar of $Q$ (Fig. 71A), which is $q$, passes through $P$ by hypothesis, and hence the polar of $P$, which is $p$, passes through $Q$, which was to be proved.

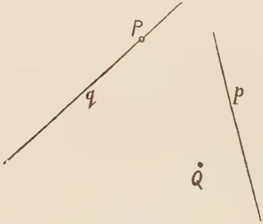



Fig. 71A.

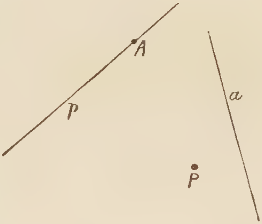



Fig. 71B.

[In the diagram, $Q$ is purposely placed *off* $p$ in order that the student may not be led to assume *from the figure* the result he has to prove.]

195. **If a line passes through a fixed point, then its pole lies on a fixed line.**

For let the fixed point be $A$ (Fig. 71B), and let its polar be $a$. Suppose $p$ is any line through $A$, and $P$ its pole. Then the polar of $P$ passes through $A$; consequently the polar of $A$ passes through $P$, *i.e.*, $a$ passes through $P$ or $P$ lies on $a$, *i.e.*, the pole of $p$ always lies on a fixed line, namely, the polar of $A$. [See note at end of last section.]

196. **Application to the geometrical construction of the polar of a point.**

The result of § 195 is important, as it affords a means of constructing the polar of any point. For through the point $A$, whether within (Fig. 72A) or outside (Fig. 72B) the curve,

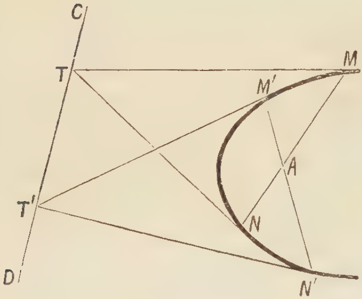



Fig. 72A.

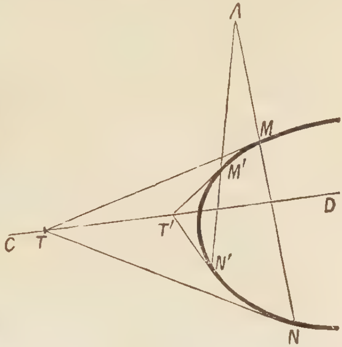



Fig. 72B.

we can certainly draw chords meeting the conic in real points—*e.g.*, $M$ and $N$, $M'$ and $N'$—and the tangents at the ends of such chords intersect at $T$, $T'$ on the polar required.

This leads to an *alternative definition of the polar.*

We first prove that as a chord turns round a fixed point the tangents at its extremities intersect on a fixed line. This line may be defined as the polar of the given point.

The advantage of this method consists in the avoidance of the use of imaginaries, but, as these enter freely into the higher parts of analytical geometry, it is desirable to see quite early how they occur.

197. **If a point moves along a fixed line, its polar passes through a fixed point.**

For it is quite clear that its polar always passes through the pole of the fixed line.

198. **Application to the geometrical construction of the pole of a given line.**

The result of § 197 gives a geometrical construction for the pole of a line $CD$ in all cases (just as the last article

did for polars), for *some* points of the line must be outside the curve, and from these, *e.g.*, $T$, $T'$, we can draw real tangents $TM$ and $TN$, and $T'M'$ and $T'N'$ to the curve, and the point of intersection of their chords of contact $MN$, $M'N'$ is the pole $A$ required. Thus it may be readily obtained by a real geometrical construction.

Similarly, this may be taken as the definition of the pole of a line, and the same remarks apply as in § 196.

199. **To find the coordinates of the pole of a given straight line.**

There are two different methods which we can use.

I. Suppose $A$ and $B$ are two points on the line. Then, by § 192, the polars of $A$ and $B$ both pass through the pole of the line $AB$, and hence their point of intersection is the pole required. Thus we select two points on the given straight line, write down the equations of the two polars, and solve them for their point of intersection.

II. The more usual method is to suppose that $(x_1, y_1)$ is the pole required, and then choose $x_1$ and $y_1$ so that the polar of the point $(x_1. y_1)$ is identical with the given line.

*Example* (i.). **To find the pole of the line $lx+my=1$ with respect to the ellipse $x^2/a^2+y^2/b^2=1$.**

Let $(x_1, y_1)$ be the pole required; then the given line must be the same as $xx_1/a^2+yy_1/b^2=1$, so that on comparison of coefficients we have

$$l=x_1/a^2, \quad m=y_1/b^2;$$

$$\therefore \quad x_1=a^2l \text{ and } y_1=b^2m.$$

Cor. If the pole is on the line, the line must be a tangent. The condition for this is $a^2l^2+b^2m^2=1$.

(ii.) **To find the condition that the lines $l_1x+m_1y=1$ and $l_2x+m_2y=1$ should be conjugate with respect to the above ellipse.**

Here the pole of the second must be on the first, *i.e.*, the point $(a^2l_2, b^2m_2)$ must be on $l_1x+m_1y=1$, and hence the condition required is

$$a^2l_1l_2+b^2m_1m_2=1.$$

(iii.) *Find the pole of the line* $6x+9y+4=0$ *with respect to the conic*

$$x^2+2xy+3y^2+2x+y+\tfrac{1}{2}=0.$$

The polar of $(x_1, y_1)$ is

$$x(x_1+y_1+1)+y(x_1+3y_1+\tfrac{1}{2})+x_1+\tfrac{1}{2}y_1+\tfrac{1}{2}=0.$$

If this be the same as $6x+9y+4=0$, we must have

$$\frac{x_1+y_1+1}{6}=\frac{x_1+3y_1+\frac{1}{2}}{9}=\frac{x_1+\frac{1}{2}y_1+\frac{1}{2}}{4},$$

and we have to solve for $x_1$ and $y_1$.

From the equality of the first and last we have

$$4x_1+4y_1+4=6x_1+3y_1+3 \quad \text{or} \quad -2x_1+y_1+1=0.$$

Similarly, from the second and last

$$4x_1+12y_1+2=9x_1+\tfrac{9}{2}y_1+\tfrac{9}{2} \quad \text{or} \quad -5x_1+\tfrac{15}{2}y_1-\tfrac{5}{2}=0.$$

Solving these two linear equations for $x_1$ and $y_1$, we find $x_1=1$ and $y_1=1$, so that the pole required is the point $(1, 1)$.

### Exercises.

Find the coordinates of the poles of the lines

11. $3x+4y=5$ with respect to $2x^2-y^2=3$.
12. $5x+y+4=0$ with respect to $x^2+2xy-y^2+2x+1=0$.
13. $x-y+12=0$ with respect to $x^2+xy+y^2=3$.

14. Find the equations of the polars of two points $P(3, 3)$, $Q(4, 4)$ with respect to $4x^2+9y^2=36$, find the coordinates of the point of intersection $R$ of the polars, and find the equation of the polar of $R$. Confirm § 192 by showing that this equation is the equation of the line joining $P$ and $Q$.

15. Prove that the coordinates of the pole of the line $lx+my=1$ with respect to the conic $Ax^2+2Hxy+By^2=1$ are

$$\frac{Bl-Hm}{AB-H^2}, \quad \frac{Am-Hl}{AB-H^2}.$$

16. Deduce that the two lines $lx+my=1$, $l_1x+m_1y=1$ are conjugate if $Bll_1-H(lm_1+l_1m)+Amm_1=AB-H^2$.

17. From the result of Ex. 16, find the condition of tangency for the line $lx+my=1$.

### 200. Polar of the centre.—Line at infinity.

The equation of the polar of the point $(x_1, y_1)$ with respect to the general conic is

$$x(ax_1+hy_1+g)+y(hx_1+by_1+f)+gx_1+fy_1+c=0.$$

Now, if $(x_1, y_1)$ be the centre of the conic, we have

$$ax_1+hy_1+g=0 \quad \text{and} \quad hx_1+by_1+f=0, \qquad (\S\ 96)$$

so that the above equation takes the form

$$(x\times 0)+(y\times 0)+n=0 \quad \text{where} \quad n=gx_1+fy_1+c,$$

and this equation does not involve either $x$ or $y$. Let us inquire into the meaning of this curious result.

In the first place, we remark that the equation of every line can be brought to the form $lx+my = 1$, and for different values of $l$ and $m$ this represents different lines. Now the intercepts on the axes of coordinates are, as is well known, $1/l$ and $1/m$ respectively, and hence, the smaller $l$ and $m$ become, the greater are the intercepts on the axes, and, consequently, the further away from the origin is the line. Ultimately, when $l$ and $m$ are both zero, both intercepts are infinite, and the line is wholly at an infinite distance. Whenever the coefficients of $x$ and $y$ are zero, we can reduce the equation to the form $lx+my = 1$ where $l$ and $m$ are zero. Hence we infer that

"When the coefficients of $x$ and $y$ in the equation of a straight line are both zero, the line is wholly at an infinite distance, and is called the *line at infinity*."

We accordingly interpret the equation of the polar of the centre by saying that

**the polar of the centre is the line at infinity;**

or, in other words, the centre is the pole of the line at infinity.

The last paragraph only corroborates and states in another form what we have already proved while dealing with asymptotes, as we can see in the following manner. An asymptote is a line which meets the curve in two points at infinity, *i.e.* it is a tangent whose point of contact is removed to an infinite distance; but the two asymptotes meet in the centre, and are therefore the tangents drawn from the centre to the curve. Hence the polar of the centre, being the line joining the points of contact of these tangents, must be at infinity.

### Illustrative Examples.

(i.) **If the polar of $T$ with respect to a conic whose centre is $C$ meet the line $CT$ in $U$, and $CT$ meet the curve in $A$, then**

$$CU \cdot CT = CA^2.$$

Take for axes $CA$ and $CB$, the conjugate diameter; then the point $T$ is $(x_1, 0)$ where $CT = x_1$, and the equation of the ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a = CA$ and $b = CB$.

The polar of $T$ is

$$\frac{xx_1}{a^2} = 1,$$

and hence, at $U$, $x = \dfrac{a^2}{x_1}$,

*i.e.* $$CU = \frac{a^2}{x_1},$$

so that $CU \cdot CT = a^2 = CA^2$.

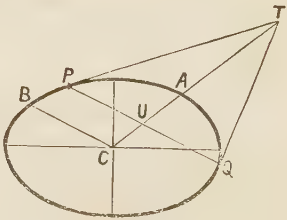



Fig. 73.

NOTE.—Since the equation of $PQ$ is $x = a^2/x_1$, $PQ$ is parallel to $BC$, whose equation is $x = 0$, *i.e.*, the polar of $T$ is parallel to the conjugate diameter. Hence, $TP$, $TQ$ being the tangents, $PQ$ is bisected by $CA$ since it is parallel to $CB$. We infer at once that "the chord of contact of the tangents drawn from any point is bisected by the line joining that point to the centre."

(ii.) *To find the locus of the poles of the tangents to one conic $A$ with respect to another conic $B$.*

This is a type of question that frequently occurs. To solve it suppose $(x_1, y_1)$ is one of the poles. Then its polar with respect to $B$ must touch $A$ by hypothesis. Thus we write down the equation of this polar, and then, putting the condition that it should touch $A$, we have a relation between $x_1$ and $y_1$ which is the equation of the locus required.

[The reader should carefully remember this method of solution.]

*E.g.*, to find the locus of the poles of the tangents of the conic $x^2/\alpha^2 + y^2/\beta^2 = 1$ with respect to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Suppose $(x_1, y_1)$ is one of the poles; then its polar is

$$x\,(ax_1 + hy_1 + g) + y\,(hx_1 + by_1 + f) + gx_1 + fy_1 + c = 0,$$

and, by hypothesis, this has to touch $x^2/\alpha^2 + y^2/\beta^2 = 1$.

Now the line $lx + my + 1 = 0$ touches this latter if $\alpha^2 l^2 + \beta^2 m^2 = 1$,

but here $$l = \frac{ax_1 + hy_1 + g}{gx_1 + fy_1 + c}, \quad m = \frac{hx_1 + by_1 + f}{gx_1 + fy_1 + c};$$

$$\therefore \quad \alpha^2 (ax_1 + hy_1 + g)^2 + \beta^2 (hx_1 + by_1 + f)^2 = (gx_1 + fy_1 + c)^2.$$

Dropping suffixes, we see that the locus is a conic whose equation is

$$\alpha^2 (ax + hy + g)^2 + \beta^2 (hx + by + f)^2 = (gx + fy + c)^2.$$

(iii.) *Find the locus of the poles of the normals to the ellipse*

$$x^2/a^2 + y^2/b^2 = 1.$$

The equation of the normal at $(x', y')$ is

$$\frac{a^2x}{x'} - \frac{b^2y}{y'} = a^2 - b^2 \quad \text{.............................} \quad (1).$$

If $(p, q)$ is the pole of this line, then the line can also be represented

by $$\frac{xp}{a^2} + \frac{yq}{b^2} = 1;$$

and therefore $$\frac{a^2}{x'}\bigg/\frac{p}{a^2} = -\frac{b^2}{y'}\bigg/\frac{q}{b^2} = a^2 - b^2,$$

whence $$x' = \frac{a^4}{p\,(a^2 - b^2)}, \quad y' = -\frac{b^4}{q\,(a^2 - b^2)}.$$

But $(x', y')$ is on the ellipse, and therefore

$$x'^2/a^2 + y'^2/b^2 = 1;$$

$$\therefore \quad \left\{\frac{a^3}{p(a^2-b^2)}\right\}^2 + \left\{-\frac{b^3}{q(a^2-b^2)}\right\}^2 = 1$$

or

$$p^2q^2(a^2-b^2)^2 = a^6q^2 + b^6p^2.$$

Therefore the locus of $(p, q)$ is

$$x^2y^2(a^2-b^2)^2 = a^6y^2 + b^6x^2.$$

## MISCELLANEOUS EXERCISES ON CHAP. XV.

18. Find, from first principles, the equation of the polar of the point (1, 1) with respect to the parabola $y^2 = 2x$.

19. Write down the equations of the polar of the point (1, 2) with respect to the conics $xy + x + 2y = 0$, $x^2 + xy + y^2 = 1$, $x^2 - y^2 - a^2$, $ax^2 + 2hxy + by^2 + 2gx = 0$.

20. Find the coordinates of the pole of the line $lx + my = 1$ with respect to (i.) $x^2/a^2 - y^2/b^2 = 1$; (ii.) $y^2 = 4ax$; (iii.) $xy = c^2$.

21. Deduce from the results of Ex. 20 the conditions that the two lines $lx + my = 1$, $l'x + m'y = 1$ should be conjugate with respect to the conics in Ex. 20.

22. Prove that the polar of a focus of an ellipse is the corresponding directrix.

23. Extend Ex. 22 to the case of a parabola.

24. Tangents are drawn from a point $T$ $(x_1, y_1)$ to the parabola $y^2 = 4ax$. Show that the coordinates of the middle point $M$ of the chord of contact are $X = \frac{y_1^2 - 2ax_1}{2a}$, $Y = y_1$.

25. Show that the line $TM$ in Ex. 24 is parallel to the axis.

26. The polar of a point $T$ with respect to a parabola meets the diameter through $T$ in $V$. Show that the middle point of $TV$ is on the curve, and that the tangent there is parallel to the polar of $T$.

27. The polars of a point with respect to an ellipse and the auxiliary circle meet on the major axis of the ellipse.

28. From points on a given straight line, pairs of tangents are drawn to a given circle. Prove that the chords of contact pass through a fixed point.

29. If a system of ellipses be described having a common axis major, prove that the polars of any fixed point with respect to them all pass through a fixed point on the major axis.

Extend the proposition to the case in which the ellipses all touch at the extremities of a common diameter.

30. Find the pole of $x\cos\alpha + y\sin\alpha - p = 0$ with reference to the parabola $y^2 = 4ax$.

31. Find the locus of $(x', y')$ when its polar with respect to $y^2 = 4ax$ is parallel to a given line $lx + my = 1$.

32. If the polar of a point with respect to $x^2/a^2 - y^2/b^2 = 1$ pass through one end of the conjugate axis of the curve, prove that the pole will lie on the tangent to the conjugate hyperbola at the other end of that axis.

33. Find the locus of the poles with regard to the circle $x^2 + y^2 = a^2$ of the tangents to the circle $x^2 + y^2 - 2ax = 0$.

34. If a straight line touches a circle whose centre is the vertex of a parabola and whose diameter is equal to the latus rectum, prove that the locus of its pole with respect to the parabola is a rectangular hyperbola.

35. Prove that the locus of the poles of the tangents to the circle $(x-b)^2 + y^2 = c^2$, taken with regard to the circle $x^2 + y^2 = a^2$, is the conic

$$(c^2 - b^2)x^2 + c^2y^2 + 2a^2bx - a^4 = 0.$$

36. Prove that the polars of a point $(f, g)$ with respect to conics whose equations are obtained by giving $k$ different values in $x^2 + y^2 + 2kxy = 1$ all intersect in a point.

If $(f, g)$ move on a fixed straight line, prove that the intersection of the polars moves on a fixed hyperbola.

37. If the polar of a point with respect to $x^2/a^2 + y^2/b^2 = 1$ touch the hyperbola whose equation is $x^2/a^2 - y^2/b^2 = 1$, the locus of the point is the hyperbola.

38. $N$ is the foot of the perpendicular let fall from a point $P$ on its chord of contact with respect to a given ellipse. Show that the locus of $N$ is a rectangular hyperbola if $P$ moves on a fixed straight line through the centre of the ellipse.

39. If $PQ$ be the line joining any point $P$ to the intersection $Q$ of the polars of $P$ with respect to two fixed circles, find the locus of the middle point of $PQ$.

40. If a straight line move so that its pole with respect to a given circle moves along a given straight line, prove that its pole with respect to any other circle will also move along a straight line.

41. If the chord of contact of two tangents to a parabola drawn from an external point be always normal to the parabola, prove that the equation of the locus of the external point is

$$y^2(x + 2a) + 4a^3 = 0.$$

42. Prove that the system of circles with respect to which a given point has the same polar line have a common radical axis, viz., the line bisecting at right angles the perpendicular from the point to its polar.

# CHAPTER XVI.

## REPRESENTATION OF POINTS ON A CONIC BY MEANS OF ONE PARAMETER.

201. **Parameters.**—In dealing with questions relating to points on a conic it is frequently more convenient to suppose such points to be given by means of one independent variable than by two coordinates connected by an equation. Thus, if we can find simple expressions for the coordinates of points on a conic in terms of one variable quantity, a point on the curve may be looked on as determined by a definite value of the variable, the variable being usually called the *parameter*.

For example, in the case of the circle $x^2+y^2=a^2$ we may put

$$x = a\cos\theta, \quad y = a\sin\theta,$$

and then to each point on the curve corresponds a different value of $\theta$ which is now the parameter.

It is clearly desirable that for every point on the curve there should be a value of the parameter, and that no two points should correspond to the same value of the parameter.

Of course, we could express both the coordinates in terms of one of them, for by solving a quadratic equation we can express $y$ in terms of $x$; but, as the expression

would usually involve surds, it would manifestly be difficult to work with. We may, however, be able to express both $x$ and $y$ in a simple manner in terms of some other variable, and in the three equations first found for the several varieties of conics we shall, in fact, be able to obtain very convenient expressions.

202. **Simple cases of parametric representation.**

I. **In the parabola $y^2 = 4ax$** if we put

we get
$$\left.\begin{aligned} y &= 2a\mu \\ x &= a\mu^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(1)},$$

so that we have expressed $x$ and $y$ very simply in terms of the parameter $\mu$ when $(x, y)$ is on the curve. Moreover, as each point on the curve has a different ordinate, each such point corresponds to a different value of $\mu$.

As $\mu$ beginning at $-\infty$ gradually changes to $+\infty$, then $y$ begins by being $-\infty$ and gradually changes to $+\infty$; so the point on the curve begins by being at an infinite distance on the portion of the curve below the axis, gradually approaches the vertex, and then moves away to an infinite distance in the portion above the axis.

The value $\mu = 0$ corresponds to the vertex, for then

$$x = 0, \quad y = 0.$$

II. **The ellipse $x^2/a^2 + y^2/b^2 = 1$.**

Here, if we put
$$\left.\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(2)},$$

the point $(x, y)$ is on the curve, for then

$$\begin{aligned} \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 &= \cos^2\theta + \sin^2\theta \\ &= 1. \end{aligned}$$

Also, as $\theta$ gradually changes from $0°$ to $360°$ the corresponding point traces out the ellipse in the counter-clockwise direction. Finally, no two points correspond to the same value of $\theta$, because, if *both* $\sin\theta$ and $\cos\theta$ are given, we obtain one value only of $\theta$ between $0°$ and $360°$.

III. **Hyperbola $x^2/a^2 - y^2/b^2 = 1$.** Here, if we put

$$x = a \sec\theta, \quad y = b \tan\theta \;\text{............}\; (3),$$

the point $(x, y)$ is on the curve, and to two different points correspond different values of $\theta$.

If $\theta$ gradually increases from $0°$ to $360°$, the corresponding point completely describes the hyperbola, but the order of description should be carefully noted. The point begins at the right-hand vertex, and moves to an infinite distance in the upper portion of the right-hand branch when $\theta = 90°$; then it starts from an infinite distance in the lower portion of the left-hand branch (for now $\sec\theta$ and $\tan\theta$ are both negative), and, gradually approaching the left-hand vertex, reaches it when $\theta = 180°$, for then $\sec\theta = -1$, $\tan\theta = 0$. Then it moves to an infinite distance in the upper portion of the left-hand branch, and finally comes back to the starting-point along the lower portion of the right-hand branch.

We may thus speak of "*the point* $\mu$" or "*the point* $\theta$" when we mean the point whose parameter is $\mu$ or $\theta$.

**Exercises.**

1. If $x = 1 + t$, $y = 1 + t^2$, $t$ being variable, show that the point $(x, y)$ describes a parabola whose vertex is at $(1, 1)$, and describe the motion of the point round the curve when $t$ varies from $-\infty$ to $+\infty$.

[To find the equation eliminate $t$.]

2. If $x = a\cos\theta + c$, $y = b\sin\theta + d$, show that $(x, y)$ describes an ellipse whose centre is at $(c, d)$, and trace the motion of the point round the curve as $\theta$ varies from 0 to $2\pi$.

3. In the rectangular hyperbola $x^2 - y^2 = a^2$ show that we may put $x = a\sec\theta$, $y = a\tan\theta$.

4. In the ellipse $x^2/a^2 + y^2/b^2 = 1$ find the parametric angles of (i.) the ends of the equi-conjugate diameters, (ii.) the ends of the latera recta.

[Find the coordinates of these points and then find $\theta$.]

203. **Application of the method in the case of the parabola.**

We shall now give some illustrations of the use of representing points on a parabola $y^2 = 4ax$ by means of

$$x = a\mu^2, \quad y = 2a\mu.$$

One of the most striking advantages arises when we re-

quire the points of intersection of the parabola with other curves whose equations are given. For, instead of having to solve two equations for $x$ and $y$, we know that the point $x = a\mu^2$, $y = 2a\mu$ is on the parabola for all values of $\mu$, *so we substitute for $x$ and $y$ in the equation of the given curve their values in terms of $\mu$, and there results an equation in $\mu$ giving the parameters of the points of intersection.*

*Example* (i.). **To find the parameters of the points of intersection of the parabola $y^2 = 4ax$ with the line $lx + my + n = 0$, and to deduce the conditions of tangency.**

Since for any point on the parabola

$$x = a\mu^2, \quad y = 2a\mu,$$

the parameters of the points of intersection are given by

$$l(a\mu^2) + m(2a\mu) + n = 0 \quad \text{or} \quad \mu^2 al + 2\mu am + n = 0,$$

which is a quadratic for $\mu$, showing that there are two intersections, for to each value of $\mu$ there answers one point on the curve.

If the line touches the parabola the roots of the quadratic must obviously be equal.

The condition for this is

$$a^2m^2 = anl \quad \text{or} \quad am^2 = ln.$$

*Example* (ii.). **A circle meets a parabola in four points, and the sum of the ordinates of these points is zero.**

For the equation of the circle is of the form

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

and accordingly, putting $x = a\mu^2$, $y = 2a\mu$, we find an equation of the fourth degree for $\mu$, viz.,

$$\mu^4a^2 + \mu^2(2ga + 4a^2) + 2\mu fa + c = 0.$$

Thus there are four points of intersection, since the equation in $\mu$ is of the fourth degree.

Since there is no term in $\mu^3$, the sum of the roots is zero.

But $2a$ times each root is the corresponding ordinate; therefore the sum of the ordinates is zero.

204. **To find the equation of the chord joining two points whose parameters are $\mu_1$ and $\mu_2$.**

The chord joining $(x_1, y_1)$ to $(x_2, y_2)$ is

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \quad \text{or} \quad \frac{y-2a\mu_1}{x-a\mu_1^2} = \frac{2a(\mu_2-\mu_1)}{a(\mu_2^2-\mu_1^2)} = \frac{2}{\mu_2+\mu_1}.$$

Hence $\quad y(\mu_2+\mu_1) - 2a\mu_1(\mu_2+\mu_1) = 2x - 2a\mu_1^2$

or $$y(\mu_2+\mu_1) - 2x = 2a\mu_1\mu_2 \quad \text{.............. (4).}$$

**205. Tangent at the point $\mu$.**—By putting $\mu_2 = \mu_1 = \mu$ in the equation of the chord, we deduce the equation of the tangent at the point whose parameter is $\mu$, namely

$$2y\mu - 2x = 2a\mu^2$$

or

$$x - \mu y + a\mu^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

206. **Geometrical meaning of the value of $\mu$.**

It is useful to know the geometrical meaning of $\mu$ in the equations $x = a\mu^2, \quad y = 2a\mu.$

We can deduce this immediately from the equation of the tangent, for the angle the tangent makes with the axis is $1/\mu$, and hence $\mu$ *represents the cotangent of the angle which the tangent at the point $\mu$ makes with the axis of the parabola.*

207. We saw in § 48 that the line

$$y = mx + a/m$$

touches the parabola for all values of $m$. It is interesting to compare this with the equation of the tangent at the point $\mu$, which is $x - \mu y + a\mu^2 = 0.$

We see at once that the two are identical if

$$\mu = 1/m.$$

We might thus have used $m$ as our parameter, and then we should have

$$x = a/m^2, \quad y = 2a/m;$$

and now $m$ denotes the *tangent* of the inclination of the tangent at the corresponding point to the axis of the curve. But by using $\mu$ we avoid fractions.

*Example.—If the tangents at two points are at right angles, the chord joining them passes through the focus.*

Let the points be $\mu_1$ and $\mu_2$; then, since the tangents are at right angles, we have $\mu_1\mu_2 = -1.$

But the chord joining them is

$$y(\mu_1 + \mu_2) - 2x = 2a\mu_1\mu_2,$$

and this meets the axis where

$$x = -a\mu_1\mu_2 = a,$$

*i.e.* in the focus.

208. **Two tangents can be drawn to a parabola from any point, and, if they are at right angles, the point is on the directrix.**

Suppose the coordinates of the point are $x_1$, $y_1$. The equation of the tangent at the point $\mu$ is $x-\mu y+a\mu^2=0$, and, if this pass through $(x_1, y_1)$, we have

$$a\mu^2-\mu y_1+x_1=0,$$

which is a quadratic for $\mu$.

There are thus two values of $\mu$, giving points on the curve the tangents at which pass through the given point. If these values are $\mu_1$, $\mu_2$, then when the tangents are at right angles we have

$$\mu_1\mu_2=-1.$$

But, by the theory of quadratics, $\mu_1\mu_2=x_1/a$; or $x_1/a=-1$, *i.e.*, $x_1+a=0$, or, in other words, the point $(x_1, y_1)$ is on the directrix.

### Exercises.

**5. Prove that the coordinates of the point of intersection of the tangents of a parabola at the points $\mu_1$ and $\mu_2$ are**

$$a\mu_1\mu_2, \quad a(\mu_1+\mu_2).$$

[This result is extremely important.]

6. Prove that, if $am^2$, $2am$ are the coordinates of one end of a focal chord of a parabola, the coordinates of the other end are

$$\frac{a}{m^2}, \quad -\frac{2a}{m}.$$

7. In the parabola $y^2=4x$, find the coordinates of the points for which the values of the parameter are $1, 2, -\frac{1}{2}, -1\frac{1}{2}$.

8. What is the value of the parameter $\mu$ for the extremities of the latus rectum of $y^2=4ax$?

9. Find the coordinates of the point on the parabola $y^2=x$ the tangent at which makes an angle 60° with the axis.

10. Obtain the equation of the chord joining two points $\mu_1$, $\mu_2$ by putting for $(x_1, y_1)$, $(x_2, y_2)$ their values in the equation

$$(y-y_1)(y-y_2)=y^2-4ax.$$

11. A chord $PQ$ of a parabola moves so that the product of the tangents of the angles that the tangents at $P$ and $Q$ make with the axis is constant. Show that these tangents meet on a fixed ordinate of the parabola.

Show also that, if the sum of the tangents of the angles be constant, the chord meets the tangent at the vertex in a fixed point.

209. **Normal at any point to the parabola.**

The normal at the point $\mu$ passes through this point, and is at right angles to the tangent. Now the tangent is

$$x - \mu y + a\mu^2 = 0;$$

consequently the normal at $\mu$ $(a\mu^2, 2a\mu)$ is

$$\mu(x - a\mu^2) + (y - 2a\mu) = 0$$

or $$\boldsymbol{\mu x + y = 2a\mu + a\mu^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(6)}.$$

This equation is practically that obtained in § 181. The student should verify this by comparing the geometrical meanings of $m$ and $\mu$.

210. **Three normals can be drawn from any point to a parabola.** (*Cf.* § 180.)

Let $(x_1, y_1)$ be the given point; then the normal at the point $\mu$ passes through this point if $\quad \mu x_1 + y_1 = 2a\mu + a\mu^3$

or $$a\mu^3 + \mu(2a - x_1) - y_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\text{A}).$$

Now this is a cubic equation for $\mu$, and, consequently, there are three roots, and, corresponding to each, a point on the curve, the normal at which passes through $(x_1, y_1)$.

**Caution.**—From the theory of equations, it is known that equation (A) must have three roots or one root real (not two real or all imaginary). To these correspond one or three real normals, the others being imaginary.

Cor.—If $\mu_1$, $\mu_2$, $\mu_3$ are the roots of the equation in $\mu$, we have

$$\mu_1 + \mu_2 + \mu_3 = 0;$$
$$\mu_2\mu_3 + \mu_3\mu_1 + \mu_1\mu_2 = (2a - x_1)/a;$$
$$\mu_1\mu_2\mu_3 = y_1/a.$$

Of these three equations, the first is the condition that the three normals should meet in a point, for it does not depend on the coordinates of the points.

It follows at once from it that, if the normals at these points be concurrent, the sum of the ordinates is zero, for this sum is

$$2a(\mu_1 + \mu_2 + \mu_3).$$

The other two equations give the coordinates of the point of intersection.

*Example (..). If the normals at two points $A$ and $B$ of a parabola intersect on the curve, then the line $AB$ passes through a fixed point.*

Suppose that the normals in the points $\mu_1$, $\mu_2$ meet in a point $P$ on the curve whose parameter is $\mu$ and whose coordinates are $x$, $y$. Then the three values of $\mu$ corresponding to points whose normals pass through $P$ are $\mu$, $\mu_1$, $\mu_2$. Hence, by the above,

$$\mu\mu_1\mu_2 = y/a = 2a\mu/a = 2\mu \quad \text{or} \quad \mu_1 u_2 = 2.$$

Now the chord joining the points $\mu_1$, $\mu_2$ ($AB$) is

$$y(\mu_1+\mu_2)-2x = 2a\mu_1\mu_2,$$

so that it meets the axis of the curve in the point

$$x = -a\mu_1\mu_2,\ y = 0;\quad \text{or}\quad (-2a,\ 0);$$

therefore the chord $AB$ passes through a fixed point on the axis produced, distant $2a$ from the vertex.

*Example* (ii.). *To find the points of the parabola* $y^2 = 4x$, *the normals at which pass through the point* $(\frac{15}{4},\ -\frac{3}{4})$.

Here $a = 1$, and we have $x = \mu^2$, $y = 2\mu$.

The normal at any point $\mu$ is $\mu x + y = 2\mu + \mu^3$.

Hence, if it passes through $(\frac{15}{4},\ -\frac{3}{4})$, we have the cubic for $\mu$, viz.

$$\mu^3+\mu(2-x)-y = 0,\quad i.e.\ \mu^3-\mu\tfrac{7}{4}+\tfrac{3}{4} = 0.$$

This being of the third degree, we cannot solve it directly; but one solution clearly is $\mu = 1$. Hence the others are given by

$$\mu^2+\mu-\tfrac{3}{4} = 0,$$

*i.e.* they are $\frac{1}{2}$, $-\frac{3}{2}$. Consequently the three points are

$\mu = 1,\ x = 1,\ y = 2$: $\mu = \frac{1}{2},\ x = \frac{1}{4},\ y = 1$; $\mu = -\frac{3}{2},\ x = \frac{9}{4},\ y = -3$.

The reader should draw a figure so as to actually see three concurrent normals.

## Exercises.

12. The equations of the normals at the ends of the latus rectum of $y^2 = 4x$ are $x+y = 3$ and $x-y = 3$, respectively.

13. If the normal at $\mu$ meet the axis in $G$, find the length $AG$, and hence show that **the subnormal $NG$ is constant and equal to the semi-latus rectum.**

14. Show that the normal at the point $\mu$ meets the curve again in the point for which the value of the parameter is $-(\mu^2+2)/\mu$.

Write down the coordinates of this point. Draw the figure for $\mu = 1$.

15. Find the points on the parabola $y^2 = 4ax$ the normals at which pass through the point $(9,\ -6)$, and draw a figure.

16. If the normals at $P$, $Q$, $R$ meet in a point, the sum of the ordinate of $P$, $Q$, $R$ is zero.

Hence, if the normals at $Q$ and $R$ meet in the normal at a fixed point $P$, show that $QR$ is parallel to a fixed straight line.

17. From the point $\mu_1$ on the parabola $y^2 = 4ax$, two normals can be drawn besides the normal at this point, and the parameters of their feet are given by $$\mu^2+\mu\mu_1+2 = 0.$$

[Put $x = a\mu_1^2$, $y = 2a\mu_1$ in cubic for $\mu$ and note that $\mu = \mu_1$ is a root.]

211. **Application to the ellipse.—Eccentric angle.**

We have seen that the coordinates of any point of the ellipse $x^2/a^2+y^2/b^2=1$ can be expressed in the form

$$x = a\cos\theta, \quad y = b\sin\theta.$$

*The geometrical meaning of* $\theta$ is very simple. Suppose, in fact, that $P$ is the point on the ellipse, corresponding to a definite value $\theta$ of the angle. Draw $PN$ perpendicular to $ACA'$ and produce $NP$ to $P'$ such that

$$P'N : PN = a : b;$$

then the coordinates of $P'$ are $x = a\cos\theta$,

$$y = \frac{a}{b}\times b\sin\theta = a\sin\theta.$$

Thus $P'$ always lies on a circle called the *auxiliary circle*, its equation being

$$x^2+y^2=a^2,$$

so that the major axis is a diameter.

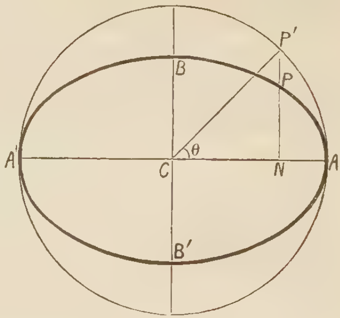



Fig. 74.

Further, since

$$CP' = a \quad \text{and} \quad CN = a\cos\theta,$$

$\theta$ is the angle $NCP'$.

Hence, *if we describe a circle on the major axis as diameter, and produce the ordinate of* $P$ *to meet this circle in* $P'$, *then the angle* $P'CA$ *is called the eccentric angle of the point* $P$, *and the coordinates of* $P$ *are* $a\cos\theta$, $b\sin\theta$, *where* $\theta$ *is the eccentric angle.*

**Corresponding points.**—Definition.—$P'$ is called the point corresponding to $P$ on the auxiliary circle. $P$ and $P'$ are sometimes called **corresponding points.**

Cor. If the coordinates of a point on the ellipse be $x$, $y$, the coordinates of the corresponding point on the auxiliary circle are $x$, $\dfrac{a}{b}y$.

The method of measuring the eccentric angle must be carefully noticed. If $P$ be a point on the ellipse and $P'$ the corresponding point on the auxiliary circle, then the eccentric angle of $P$ is the angle $ACP'$ between the positive direction of the major axis and the radius $CP'$ of the auxiliary circle.

212. **To show that the area of an ellipse whose semi-axes are $a$ and $b$ is $\pi ab$.**

Suppose the major axis divided into a very large number of equal parts. Let $NN'$ be one of these parts, and draw ordinates $P'PN$, $Q'QN'$ to meet the ellipse in $P$, $Q$, and the auxiliary circle in $P'$, $Q'$. Complete the rectangles $NN'QR$ and $NN'Q'R'$, and follow out the same process with respect to every portion of $AA'$.

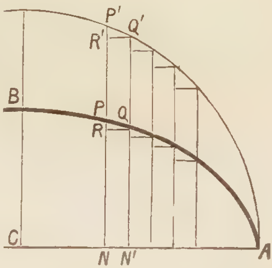


Fig. 75.

Since

$$QN' : Q'N' = b : a,$$

$$\therefore \text{ rect. } NN'QR : \text{rect. } NN'Q'R' = b : a.$$

Similarly, all other such rectangles are in the same ratio. Thus the sum of the rectangles terminated by the ellipse bears to the sum of those terminated by the circle the ratio $b : a$.

But, when the number of parts is made indefinitely large, the sum of the rectangles of the type $NN'QR$ becomes the area of the upper half of the ellipse, and the sum of those of the type $NN'Q'R'$ becomes the area of the upper half of the circle. Hence

$$\text{area of ellipse} : \text{area of circle} = b : a.$$

But the area of the circle is $\pi a^2$.

Hence the **area of the ellipse** $= \pi a^2 . \dfrac{b}{a}$

$$= \pi ab \quad \ldots\ldots \quad (7).$$

*Example.—To find the area of the ellipse* $Ax^2+2Hxy+By^2=1$.

We need to find the product of the semi-axes.

Now the semi-axes $a$, $b$ are given by the equation in $r$

$$\left(A-\frac{1}{r^2}\right)\left(B-\frac{1}{r^2}\right)=H^2 \quad\text{or}\quad \frac{1}{r^4}-\frac{1}{r^2}(A+B)+AB-H^2=0.$$

Thus $$\frac{1}{a^2b^2}=AB-H^2; \qquad (\textit{Tut. Alg.}, \text{ II., § 156})$$

so $$\text{the area}=\pi ab=\frac{\pi}{\sqrt{AB-H^2}}.$$

[The expression under the radical is positive, since $AB-H^2>0$ for an ellipse.]

**Exercises.**

18. Find the coordinates of the points of intersection of the normals to the parabola $y^2=4ax$ at the points $\mu$, $\mu'$ in the form

$$x=2a+a(\mu^2+\mu\mu'+\mu'^2), \quad y=-a\mu\mu'(\mu+\mu').$$

19. If the point $\mu$ approach, and ultimately coincide with, the point $\mu'$, show that the point of intersection of the normals is

$$x=2a+3a\mu^2, \quad y=-2a\mu^3.$$

This, the point of intersection of two consecutive normals, is called the **centre of curvature** at the point $\mu$.

20. The area enclosed by any two ordinates and the ellipse is to the area enclosed by the same two ordinates and the auxiliary circle as $b:a$.

21. If $P$, $Q$, $R$ be three points on the ellipse and $P'$, $Q'$, $R'$ the corresponding points on the auxiliary circle, then

$$\triangle PQR : \triangle P'Q'R' = b : a.$$

## 213. Equation of the chord of an ellipse joining two points whose eccentric angles are $\alpha$ and $\beta$.

The coordinates of the points are

$$(a\cos\alpha,\ b\sin\alpha), \quad (a\cos\beta,\ b\sin\beta).$$

Therefore the equation of the line joining the points is

$$\frac{x-a\cos\alpha}{y-b\sin\alpha}=\frac{a(\cos\alpha-\cos\beta)}{b(\sin\alpha-\sin\beta)}$$

or $$bx(\sin\alpha-\sin\beta)-ay(\cos\alpha-\cos\beta) -ab(\sin\alpha\cos\beta-\cos\alpha\sin\beta)=0,$$

or $$bx.2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}+ay.2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} -ab\sin(\alpha-\beta)=0;$$

and hence, on dividing across by $2\sin\frac{\alpha-\beta}{2}$, we find

$$bx \cos\frac{\alpha+\beta}{2} + ay \sin\frac{\alpha+\beta}{2} - ab \cos\frac{\alpha-\beta}{2} = 0$$

or $$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2} \quad \ldots\ldots \text{(8)}.$$

COR. **The equation of the tangent at the point $\alpha$ is**

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1 \quad \ldots\ldots\ldots\ldots \text{(9)}.$$

This is obtained by making $\alpha = \beta$ in the equation of the chord joining the points $\alpha, \beta$, which then becomes the tangent at the point $\alpha$.

214. The equation of the chord joining the points whose eccentric angles are $\alpha+\beta$, $\alpha-\beta$ is

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha+\beta+\alpha-\beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta+\alpha-\beta) = \cos\tfrac{1}{2}(\alpha+\beta-\alpha+\beta),$$

or $$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = \cos\beta,$$

a very simple form.

Again, if $\beta$ be constant in this equation, the chord is the tangent at the point $\alpha$ on the ellipse, whose axes are $a\cos\beta$, $b\cos\beta$, so that chords joining two points whose eccentric angles differ by a constant quantity always touch an ellipse.

COR.—If the points $\frac{1}{2}(\alpha+\beta)$, $\frac{1}{2}(\alpha-\beta)$ be the ends of a focal chord, the equation of the chord is satisfied by $x = \pm ae$, $y = 0$, and we have $\cos\beta = \pm e\cos\alpha$.

215. **The product of the perpendiculars from the focus on the tangent to an ellipse is equal to the square of the minor axis.**

The foci are $(ae, 0)$, $(-ae, 0)$; the perpendiculars from them on the tangent at the point $\alpha$ or $x\cos\alpha/a + y\sin\alpha/b = 1$ are

$$\frac{\dfrac{ae\cos\alpha}{a} - 1}{\sqrt{\dfrac{\cos^2\alpha}{a^2} + \dfrac{\sin^2\alpha}{b^2}}} \quad\text{and}\quad \frac{-\dfrac{ae\cos\alpha}{a} - 1}{\sqrt{\dfrac{\cos^2\alpha}{a^2} + \dfrac{\sin^2\alpha}{b^2}}}.$$

The product is

$$\frac{1-e^2\cos^2\alpha}{b^2\cos^2\alpha + a^2\sin^2\alpha}a^2b^2 = \frac{1-e^2\cos^2\alpha}{a^2(1-e^2)\cos^2\alpha + a^2\sin^2\alpha}a^2b^2$$
$$= \frac{1-e^2\cos^2\alpha}{a^2(1-e^2\cos^2\alpha)}a^2b^2 = b^2.$$

**216. If $CP$, $CD$ be a pair of conjugate semi-diameters of an ellipse, both above or both below the axis of $x$, then the eccentric angles of $P$ and $D$ differ by a right angle.**

If $\alpha$, $\beta$ are the eccentric angles of $P$ and $D$, then the equations of $CP$, $CD$ are

$$y = x\frac{b \sin \alpha}{a \cos \alpha} \quad \text{and} \quad y = x\frac{b \sin \beta}{a \cos \beta}.$$

But, if the lines $y = mx$ and $y = m'x$ are conjugate diameters, $mm' = -b^2/a^2$. (Ex. 6, p. 175)

Hence
$$\frac{b^2}{a^2}\frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta} = -\frac{b^2}{a^2};$$

$$\therefore \quad \cos \alpha \cos \beta + \sin \alpha \sin \beta = 0 \quad \text{or} \quad \cos(\alpha - \beta) = 0.$$

Therefore $\alpha - \beta$ is one right angle or three right angles.

Since $CP$ and $CD$ are *both* above or *both* below the axis of $x$, the difference of three right angles is inadmissible.

Cor. If the conjugate diameters be $CP$, $CD$, then, if $P$ be $\alpha$, $D$ may be taken to be $\alpha + \frac{\pi}{2}$.

**217. The sum of the squares of two conjugate semi-diameters of an ellipse is constant, and the parallelogram constructed on them as adjacent sides is also constant in area.**

By § 216, Cor., the coordinates of $P$ and $D$ are

$$a \cos \alpha,\ b \sin \alpha \quad \text{and} \quad a \cos\left(\alpha + \frac{\pi}{2}\right),\ b \sin\left(\alpha + \frac{\pi}{2}\right),$$

or
$$a \cos \alpha,\ b \sin \alpha \quad \text{and} \quad -a \sin \alpha,\ b \cos \alpha.$$

Hence

$$CP^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha, \quad CD^2 = a^2 \sin^2 \alpha + b^2 \cos^2 \alpha;$$

leading at once to $CP^2 + CD^2 = a^2 + b^2$.

Again, the parallelogram $= 2 \triangle CPD$

$$= 2 \cdot \tfrac{1}{2} \{a \cos \alpha\, b \cos \alpha - b \sin \alpha\, (-a \sin \alpha)\}$$

$$= ab(\cos^2 \alpha + \sin^2 \alpha) = ab. \qquad \text{(Pt. I., § 4, Cor.)}$$

Thus both results are proved.

### Exercises.

22. Show that the eccentric angles corresponding to the equal conjugate diameters are 45° and 135°.

23. In the ellipse $x^2/a^2+y^2/b^2=1$ find the equation of the chord joining the points whose eccentric angles are 30° and 120°; also of that joining the points whose eccentric angles are 30° and 60°.

24. Show that the line joining the extremities of two conjugate diameters of $x^2/a^2+y^2/b^2=1$ touches $x^2/a^2+y^2/b^2=2$.

25. Find the coordinates of the points of intersection of the tangents at $\alpha$, $\beta$ in the form

$$x=a\cos\tfrac{1}{2}(\alpha+\beta)/\cos\tfrac{1}{2}(\alpha-\beta), \quad y=b\sin\tfrac{1}{2}(\alpha+\beta)/\cos\tfrac{1}{2}(\alpha-\beta).$$

If $(\alpha+\beta)$ be constant, show that the locus of $(x, y)$ is a straight line through the centre, and, if $(\alpha-\beta)$ be constant, the locus is an ellipse whose equation is $x^2/a^2+y^2/b^2=\sec^2\frac{1}{2}\gamma$ where $\alpha-\beta=\gamma$.

26. If the eccentric angle of $P$ be $\theta$, show that, with the usual notation,

$$CD^2=a^2(1-e^2\cos^2\theta), \quad SP=a(1-e\cos\theta), \quad S'P=a(1+e\cos\theta),$$

and deduce that $$SP.S'P=CD^2.$$

27. The portion of a latus rectum intercepted between the auxiliary circle and the ellipse is equal to the minor axis.

28. The perpendicular $p$ on a tangent to an ellipse makes an angle $\alpha$ with the major axis. If $P$ be the point whose eccentric angle is $\alpha$, show that $CP=p$.

[Remember that $p^2=a^2\cos^2\alpha+b^2\sin^2\alpha$.]

29. Deduce the equation of the chord joining the points $\alpha$, $\beta$ from the form $$\frac{(x-x_1)(x-x_2)}{a^2}+\frac{(y-y_1)(y-y_2)}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1.$$

30. $P$ and $Q$ are two points on an ellipse, and $p$, $q$ the corresponding points on the auxiliary circle. If the equation of $PQ$ be $lx+my=1$, show that the equation of $pq$ is

$$lx+m\frac{a}{b}y=1.$$

[Find the equations of both in terms of the eccentric angles of $P$ and $Q$.]

Hence show that $PQ$, $pq$ meet on the major axis of the ellipse.

31. **Prove that the tangent at any point of an ellipse and the tangent at the corresponding point of the auxiliary circle meet on the major axis.**

**218. To find the equation to the normal at any point of an ellipse.**

The normal at the point $\theta$ is perpendicular to the tangent at that point and passes through the point.

But the equation of the tangent is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1;$$

therefore the normal is

$$\frac{a}{\cos\theta}(x - a\cos\theta) = \frac{b}{\sin\theta}(y - b\sin\theta)$$

or
$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad \text{............} \quad \mathbf{(10)},$$

which can also be deduced at once from the equation of the normal previously given.

219. **Four normals can be drawn from any point to an ellipse.**

If the normal at the point $\theta$ passes through a given point $(x_1, y_1)$, we have
$$\frac{ax_1}{\cos\theta} - \frac{by_1}{\sin\theta} = a^2 - b^2.$$

This is an equation for $\theta$, from which we can determine all the points on the conic the normals at which pass through $(x_1, y_1)$.

Now, to solve an equation containing both $\cos\theta$ and $\sin\theta$ the common device is to put
$$\tan\frac{\theta}{2} = t,$$

so that
$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2};$$

and hence, putting in these values, we find an equation for $t$.

In our case we get

$$\frac{ax_1(1+t^2)}{1-t^2} - \frac{by(1+t^2)}{2t} = (a^2 - b^2)$$

or
$$ax_1 2t(1+t^2) - by_1(1-t^4) = (a^2-b^2)2t(1-t^2).$$

That is, $\quad t^4 . by_1 + 2t^3(ax_1 + a^2 - b^2) + 2t(ax_1 - a^2 + b^2) - by_1 = 0. \qquad$ (B)

This equation, being of the fourth degree, has four roots; each root gives a value of $\cos\theta$ and $\sin\theta$, and hence we see that through any point four normals can be drawn to an ellipse.

**Caution.**—Equation (B) can have imaginary roots. From the theory of equations it is known that it has either 4 or 2 real roots, and hence to these three cases correspond 4 or 2 real normals, the others being imaginary.

220. From the last article the reader will have seen the advantage of putting $t = \tan \frac{1}{2}\theta$ in discussing some problems connected with the eccentric angle. With this value of $t$ we have

$$x = a\cos\theta = a\frac{1-t^2}{1+t^2}, \quad y = b\sin\theta = b\frac{2t}{1+t^2};$$

so that we have represented the coordinates of any point of the curve *rationally* in terms of one parameter $t$. This is the secret of the simplification obtained by introducing $t$, for rational functions are the ones whose properties are best known. The same remarks apply to the hyperbola.

221. **Application to the hyperbola.**—In the hyperbola $x^2/a^2 - y^2/b^2 = 1$, the coordinates of any point can be expressed in terms of one parameter $\theta$, thus:

$$x = a\sec\theta, \quad y = b\tan\theta. \qquad (\S\ 202)$$

*The geometrical meaning of $\theta$.*—Let $P$ be the point whose coordinates are $a\sec\theta$, $b\tan\theta$. Draw the ordinate $PN$, and from $N$ draw a tangent $NT$ to the auxiliary circle, and join $CT$. Then

$$\sec TCN = CN/CT$$
$$= x/a = \sec\theta.$$

Therefore the angle $TCN$ is equal to the parametric angle $\theta$ of the point $P$.

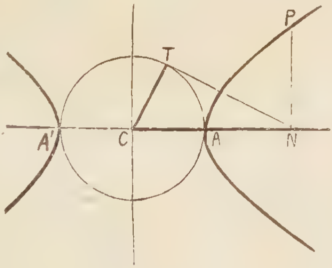



Fig. 76.

*Properties of conjugate diameters of a hyperbola.*—Some of these properties can, by the aid of the angle $\theta$, be obtained more easily than in the method indicated in § 167.

If $CP$ be a diameter and $CD$ its conjugate, we have

coordinates of $P$ are

$a\sec\theta$, $b\tan\theta$;

and the

coordinates of $D$ are

$a\tan\theta$, $b\sec\theta$.

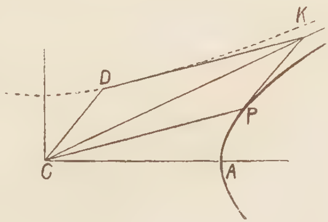



Fig. 77.

(i.) $\therefore\ CP^2 - CD^2 = (a^2 \sec^2\theta + b^2 \tan^2\theta) - (a^2 \tan^2\theta + b^2 \sec^2\theta)$
$= (a^2 - b^2)(\sec^2\theta - \tan^2\theta)$
$= a^2 - b^2.$

(ii.) If parallelogram $CPKD$ be completed,

$$\text{area of } CPKD = 2\Delta CPD$$
$$= ab(\sec^2\theta - \tan^2\theta) \qquad \text{(Pt. I., § 4, Cor.)}$$
$$= ab.$$

(iii.) The tangents $PK$, $DK$ are, respectively,

$$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1,$$
$$\frac{x \tan\theta}{a} - \frac{y \sec\theta}{b} = -1.$$

Their point of intersection $K$ satisfies the equation

$$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} + \frac{x \tan\theta}{a} - \frac{y \sec\theta}{b} = 0$$

or $$\frac{x}{a} = \frac{y}{b}, \quad \text{which is the asymptote.}$$

Therefore $K$ lies on the asymptote.

## 222. Normals from any point to a hyperbola.

(i.) In a manner similar to § 218, it can be shown that the normal at the point $\theta$ of a hyperbola is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 \quad \dots\dots\dots\dots \quad (\mathbf{11}).$$

(ii.) Thus the points the normals at which pass through $(x_1, y_1)$ are given by

$$\frac{ax_1}{\sec\theta} + \frac{by_1}{\tan\theta} = a^2 + b^2.$$

To solve this equation, put

$$\tan \tfrac{1}{2}\theta = t;$$

then $$\sec\theta = \frac{1+t^2}{1-t^2}, \quad \tan\theta = \frac{2t}{1-t^2},$$

and the equation becomes

$$by_1 t^4 + 2t^3\{a^2 + b^2 + ax_1\} + 2t\{a^2 + b^2 - ax_1\} - by_1 = 0.$$

From this equation we see that, just as with the ellipse, four normals can be drawn from any point to a hyperbola.

223. We have seen how, in the case of conics, the coordinates of any point on the curve can be expressed as functions of one variable called the parameter. Conversely, when the coordinates of a point are expressed in terms of one unknown quantity, the point lies on a curve whose equation is found by eliminating this parameter.

*Example.—If*

$$x = A\cos\theta + B\sin\theta + C, \quad y = A'\cos\theta + B'\sin\theta + C,$$

*show that* $(x, y)$ *lies on a fixed conic.*

Here we have to eliminate $\theta$ between

$$A\cos\theta + B\sin\theta + C - x = 0 \quad\text{and}\quad A'\cos\theta + B'\sin\theta + C' - y = 0.$$

Regarding $\cos\theta$ and $\sin\theta$ temporarily as two distinct unknown quantities, and solving, we get

$$\frac{\cos\theta}{\{B(C'-y) - B'(C-x)\}} = \frac{\sin\theta}{\{A'(C-x) - A(C'-y)\}} = \frac{1}{AB' - A'B};$$

and hence, since $\cos^2\theta + \sin^2\theta = 1$,

$$\{B(C'-y) - B'(C-x)\}^2 + \{A'(C-x) - A(C'-y)\}^2 = (AB' - A'B)^2,$$

an equation of the second degree, so that the locus is a conic.

The terms of the second degree are

$$(By - B'x)^2 + (Ay - A'x)^2,$$

and the factors of this are imaginary, or $ab > h^2$ in the standard notation; therefore the curve is an ellipse.

## Exercises.

32. Find the coordinates of the point of intersection of the normals at the points whose eccentric angles are $\theta$, $\phi$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the form

$$x = \frac{a^2-b^2}{a}\cos\theta\cos\phi\,\frac{\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}, \quad y = -\frac{a^2-b^2}{b}\sin\theta\sin\phi\,\frac{\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}.$$

33. If the point $\phi$ approach and ultimately coincide with $\theta$, show that the point of intersection of the normals becomes

$$x = \frac{a^2-b^2}{a}\cos^3\theta, \quad y = -\frac{a^2-b^2}{b}\sin^3\theta.$$

This, the point of intersection of two consecutive normals at $\theta$, is called the **centre of curvature** at that point.

34. Where are the feet of the four normals drawn to an ellipse from its centre?

35. Show that, from a point $(\xi, 0)$ on the major axis of an ellipse, two normals (besides the two directions of the major axis) can be drawn; and find the equation for $\tan\frac{1}{2}\theta$ in the form

$$t^2(a\xi + a^2 - b^2) + (a\xi - a^2 + b^2) = 0.$$

36. Show that the line $PD$ in Fig. 77 has a direction independent of the position of $P$.

37. From a point $(0, \eta)$ on the minor axis of an ellipse, two normals can be drawn besides the two directions of the minor axis. Show that the equation for $t$ has two roots $\pm 1$ corresponding to the ends of the minor axis, and that the remaining roots are given by

$$b\eta\,(t^2+1)+2\,(a^2-b^2)\,t = 0.$$

Thus show that the two remaining normals are only real provided $\eta$ is numerically less than $(a^2-b^2)/b$.

## Illustrative Examples.

(i.) **The orthocentre of the triangle formed by three tangents to a parabola is on the directrix.**

Take the parabola to be $y^2 = 4ax$, and suppose the triangle $PQR$ formed by the tangents at $\mu_1, \mu_2, \mu_3$. We have to form the equations of the perpendiculars from $P$, $Q$, $R$ on the opposite sides.

Now $R$ is the point of intersection of the tangents at $\mu_1$ and $\mu_2$, and its coordinates are therefore

$$a\mu_1\mu_2, \quad a\,(\mu_1+\mu_2). \qquad \text{(Ex. 5, p. 225)}$$

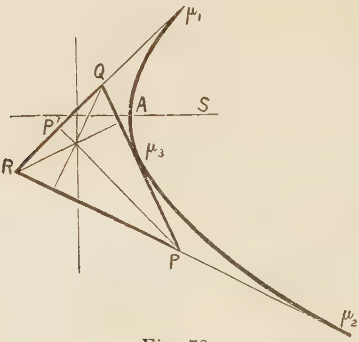



Fig. 78.

The perpendicular through it to $PQ$ or $x - \mu_3 y + a\mu_3^2 = 0$ is therefore

$$\mu_3\,(x-a\mu_1\mu_2)+(y-a\,\overline{\mu_1+\mu_2}) = 0.$$

This meets the directrix where $x = -a$, and hence

$$y = a\,(\mu_1+\mu_2)-\mu_3\,(-a-a\mu_1\mu_2) \quad \text{or} \quad y = a\left\{\mu_1+\mu_2+\mu_3+\mu_1\mu_2\mu_3\right\}.$$

But this is symmetrical in $\mu_1$, $\mu_2$, $\mu_3$.

Therefore the other perpendiculars meet the directrix in the same point.

(ii.) *If* $x = \dfrac{t^2+1}{(t-1)(t-2)}, \quad y = \dfrac{t+1}{(t-1)(t-2)},$

*show that* $(x, y)$ *lies on a hyperbola, and find the asymptotes.*

By multiplying up, we get two quadratics for $t$, and therefore, on elimination in the usual way, we find an equation of the second degree in $x$ and $y$.

To find the asymptotes, we might proceed from the equation found, but the following is more instructive.

For a point at infinity, $x$ and $y$ are infinite, hence $t = 1$ or $t = 2$; these values give the points at infinity on the curve.

Now, if $lx + my + 1 = 0$ be an asymptote, it meets the curve in two points at infinity.

Putting in for $x$ and $y$ their values in terms of $t$, we have

$$l(t^2+1) + m(t+1) + (t-1))(t-2) = 0,$$

giving the two values of $t$ for the points of intersection. If, then, $lx + my + 1 = 0$ be an asymptote, the roots of this quadratic must be either 1, 1 or 2, 2.

The quadratic is

$$t^2(l+1) + t(m-3) + l + m + 2 = 0;$$

if it be the same as $t^2 - 2t + 1 = 0$, which has roots 1, 1, we have

$$\frac{l+1}{1} = \frac{m-3}{-2} = \frac{l+m+2}{1},$$

from which we find $l = 1$, $m = -1$, so that one asymptote is

$$x - y + 1 = 0.$$

The other is given by

$$\frac{l+1}{1} = \frac{m-3}{-4} = \frac{l+m+2}{4},$$

leading to $l = -\frac{3}{7}$, $m = \frac{5}{7}$, so that the second asymptote is

$$-\tfrac{3}{7}x + \tfrac{5}{7}y + 1 = 0 \quad \text{or} \quad 3x - 5y - 7 = 0.$$

(iii.) *To find the equation of the circumcircle of the triangle formed by the tangents to a parabola at* $\mu_1, \mu_2, \mu_3$, *and to show that this circle passes through the focus.*

The equations of the tangents are

$$x - \mu_1 y + a\mu_1^2 = 0,\ x - \mu_2 y + a\mu_2^2 = 0,\ x - \mu_3 y + a\mu_3^2 = 0.$$

Consider the equation

$$A_1(x-\mu_2 y + a\mu_2^2)(x-\mu_3 y + a\mu_3^2) + A_2(x-\mu_3 y + a\mu_3^2)(x-\mu_1 y + a\mu_1^2) + A_3(x-\mu_1 y + a\mu_1^2)(x-\mu_2 y + a\mu_2^2) = 0,$$

in which the $A$'s are numerical factors. Since it is of the second degree, it represents a conic, and it is easy to see that each term of the

sum vanishes for the point of intersection of two of the tangents. Therefore for all values of the $A$'s it represents a conic circumscribing the triangle. We have only put down the conditions for a circle, viz., the coefficients of $x^2$ and $y^2$ are equal, and that of $xy$ is zero.

Thus we get

$$A_1(1-\mu_2\mu_3)+A_2(1-\mu_3\mu_1)+A_3(1-\mu_1\mu_2) = 0,$$

$$A_1(\mu_2+\mu_3)+A_2(\mu_3+\mu_1)+A_3(\mu_1+\mu_2) = 0,$$

two equations to solve for the ratios of the $A$'s. On solving in the usual way, we find

$$A_1 : A_2 : A_3 = (\mu_2-\mu_3)(1+\mu_1^2) : (\mu_3-\mu_1)(1+\mu_2^2) : (\mu_1-\mu_2)(1+\mu_3^2),$$

and the equation of the circle can be written down at once. The reader should see that the factors $(\mu_1-\mu_3)$, $(\mu_3-\mu_1)$, $(\mu_1-\mu_2)$ divide across, and that the equation reduces to

$$x^2+y^2-ax(1+\mu_2\mu_3+\mu_3\mu_1+\mu_1\mu_2)-ay(\mu_1+\mu_2+\mu_3-\mu_1\mu_2\mu_3)$$
$$+a^2(\mu_1\mu_3+\mu_3\mu_1+\mu_1\mu_2) = 0.$$

It follows at once that the circle passes through the point $(a, 0)$, *i.e.*, the focus of the parabola.

## MISCELLANEOUS EXERCISES ON CHAP. XVI.

38. Find the parameters of the points of intersection of the curve $y^2 = 4ax$ with the straight line $y = mx + c$. Deduce the condition of tangency.

39. If a chord meets the axis of a parabola in a fixed point, the product of the ordinates at its extremities is constant.

40. The sum of the ordinates of the extremities of each chord of a system of parallel chords is the same for all.

41. If the tangents drawn from a point $(x_1, y_1)$ to the curve $y^2 = 4ax$ make angles $\theta$, $\phi$ with the axis, then show that

$$\tan\theta\tan\phi = \frac{a}{x_1}, \quad \tan\theta+\tan\phi = \frac{y_1}{x_1}.$$

42. From Ex. 41 deduce that the locus of the intersection of tangents to a parabola meeting at a constant angle $\alpha$ is a conic.

$[\theta-\phi = \alpha.]$

43. Deduce the equation of the chord of an ellipse joining two points whose eccentric angles are $\alpha$ and $\beta$, by substituting in the chord equation

$$(x-x_1)(x-x_2)/a^2+(y-y_1)(y-y_2)/b^2 = x^2/a^2+y^2/b^2-1.$$

44. Prove that the product of the perpendicular distances of the focus from any two parallel tangents to an ellipse is constant.

45. $CT$ is the central perpendicular on the tangent to an ellipse at the point whose eccentric angle is $\theta$, and $CP$ is the central radius which is inclined to the axis major at an angle $\theta$. Prove that $CT = CP$.

46. If the tangents to an ellipse at the points whose eccentric angles are $\alpha$ and $\beta$ meet in the point $(x', y')$, prove that

$$\tan^2 \tfrac{1}{2}(\alpha - \beta) = x'^2/a^2 + y'^2/b^2 - 1.$$

47. The normal at the point whose eccentric angle is $\phi$ makes an angle $\psi$ with the major axis. Prove that $a \tan \phi = b \tan \psi$.

48. The normal at a point $(a \cos \phi,\ b \sin \phi)$ on an ellipse makes an angle $\theta$ with the central radius vector of the point. Prove that

$$2ab \tan \theta = (a^2 - b^2) \sin 2\phi.$$

49. The coordinates of any point on a certain curve are expressed in terms of a single variable $t$, as follows:—

$$x = 1 + t + t^2, \quad y = 1 - t + t^2.$$

Determine the nature of the curve, and trace it.

50. The length of the chord joining the points whose eccentric angles are $\alpha + \beta$, $\alpha - \beta$ is $DD' \sin \beta$, where $DD'$ is the parallel diameter.

51. If $p$ be the point on the auxiliary circle corresponding to a point $P$ of the ellipse $x^2/a^2 + y^2/b^2 = 1$, prove that the normals at $P$ and $p$ meet on a fixed circle.

52. Prove that, whatever value $\theta$ may have, the locus of a point whose coordinates can be expressed in the form

$$x = a\frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}, \quad y = \frac{2b}{e^\theta + e^{-\theta}}$$

is an ellipse.

53. In the ellipse $x^2/a^2 + y^2/b^2 = 1$ express the condition that the tangents at $\theta$, $\phi$ should be at right angles. If $t_1 = \tan \frac{1}{2}\theta$, $t_2 = \tan \frac{1}{2}\phi$, show that the condition is $b^2(1 - t_1^2)(1 - t_2^2) + 4a^2 t_1 t_2 = 0$.

54. If the tangents to an ellipse at the points $\alpha$, $\beta$ meet in $(x_1, y_1)$,

show that $$x_1 = a\frac{\cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)}, \quad y_1 = b\frac{\sin \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)};$$

also show that in the same case $\tan \frac{1}{2}\alpha$, $\tan \frac{1}{2}\beta$ are the roots of the

equation in $t$ $$t^2\left(1 + \frac{x_1}{a}\right) - 2t\frac{y_1}{b} + 1 - \frac{x_1}{a} = 0.$$

55. By using 53 and 54 show that, if the tangents drawn from $(x_1, y_1)$ to the ellipse are at right angles, then $x_1^2 + y_1^2 = a^2 + b^2$, or $(x_1, y_1)$ lies on the director circle.

56. If $P$, $Q$ are two points on the ellipse whose eccentric angles are $\phi_1$, $\phi_2$, prove that the tangents at $P$ and $Q$ will intersect on the auxiliary circle, provided

$$a^2 \cos^2 \frac{\phi_1 - \phi_2}{2} = a^2 \cos^2 \frac{\phi_1 + \phi_2}{2} + b^2 \sin^2 \frac{\phi_1 + \phi_2}{2}.$$

57. Prove also that in this case (Ex. 56), if $P'$ and $Q'$ be points on the auxiliary circle corresponding to $P$ and $Q$, the chord $P'Q'$ will touch the ellipse in the point corresponding to the point of intersection of the tangents at $P$ and $Q$.

58. If the length of the portion of the tangent to an ellipse intercepted between its axes be equal to the sum of its semi-axes, prove that the length of the perpendicular on the tangent from the centre is a mean proportional to the semi-axes.

59. If the coördinates of any point on a curve are given by the equations $x = a \tan(\theta + \alpha)$, $y = b \tan(\theta + \beta)$, where $\theta$ is variable, prove that the curve is a hyperbola, and find the position of its asymptotes.

60. Show that the focal radius to the point of intersection of two tangents to a parabola is perpendicular to the line joining the focus and the point in which the corresponding chord meets the directrix.

61. $A$ and $B$ are two points on the conic $x^2/a^2 + y^2/b^2 = 1$ such that three times the eccentric angle of the one is equal to the supplement of the other. Show that the locus of the pole of $AB$ is $a^2/x^2 + b^2/y^2 = 4$.

62. Show that the equation of the chord of the ellipse $x^2/a^2 + y^2/b^2 = 1$ joining points whose eccentric angles are $\alpha + \beta$, $\alpha - \beta$ is

$$x \cos \alpha/a + y \sin \alpha/b = \cos \beta.$$

Prove that the area of the triangle formed by this chord and the tangents at its extremities $= ab \sin^3 \beta/\cos \beta$.

63. If the coordinates $(x, y)$ of any point on a curve are given by the equations $x = ae^{2\theta} + be^{-\theta}$, $y = a'e^{2\theta} + b'e^{-\theta}$, find the equation of the curve.

64. From $C$, the centre of an ellipse (whose semi-axes are $a$, $b$) a line $CN$ is drawn perpendicular to $PN$, the normal at any point $P$ on the ellipse, and $PN$ is produced to $T$ so that $PN.PT = ab$. Prove that the locus of $T$ is a circle, centre $C$ and radius $a - b$.

65. Concentric ellipses (the common centre being at the origin) of equal area $\pi c^2$ are drawn whose principal axes are along the axes of coordinates (supposed rectangular). Show that the equation of the locus of points on these ellipses at which the tangents make an angle $\alpha$ with the axis of $x$ is

$$xy(x - y \cot \alpha)^2 + c^4 \cot \alpha = 0.$$

66. Tangents are drawn from points on the ellipse $x^2/a^2 + y^2/b^2 = 1$ to the circle $x^2 + y^2 = r^2$. Show that the chords of contact touch the ellipse $a^2x^2 + b^2y^2 = r^4$.

If $1/r^2 = 1/a^2 + 1/b^2$, show that the lines joining the centre to the points of contact with the circle are conjugate diameters of the second ellipse.

# CHAPTER XVII.

## POLAR EQUATION OF A CONIC, THE FOCUS BEING THE POLE.

In many questions relating to conics, particularly those concerning one focus, it is very convenient to use polar coordinates with the focus as pole.

In this chapter we shall obtain the equations of conics in such a system of coordinates, and discuss some of the easier results derivable from them.

**224. To find the polar equation of a conic, the focus being the pole and the axis through it the initial line.**

Let $S$ be the focus, $SX$ the perpendicular on the directrix, and $e$ the eccentricity. We shall take the initial line in the direction $SX$.

If $P$ be a point on the curve and $PN$, $PK$ be drawn perpendicular to $SX$ and the directrix respectively, we have
$$SP = r \quad \text{and} \quad \angle PSX = \theta;$$
accordingly, we have to find a relation between these two quantities.

Now
$$\begin{aligned} SP &= e.PK = e.NX \\ &= e(SX - SN) \\ &= e(SX - SP\cos\theta) \end{aligned}$$
or
$$r = e.SX - er\cos\theta.$$

Consequently,
$$r(1 + e\cos\theta) = e.SX;$$
$$\therefore \quad r = \frac{e.SX}{1 + e\cos\theta},$$
which is the relation required.

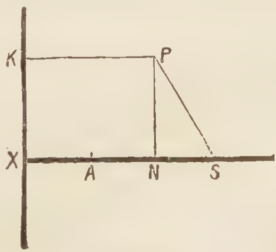



Fig. 79.

We see at once from this equation that $e.SX$ is the value of $r$ when $\theta = 90°$ (for then $\cos\theta = 0$), that is, $e.SX$ is equal to the semi-latus rectum (§ 62). If, as usual, we denote this by $l$, the equation is

$$r = \frac{l}{1 + e\cos\theta}$$

or

$$\frac{l}{r} = 1 + e\cos\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which is the usual form of the required polar equation.

Cor. If the initial line makes an angle $\alpha$ with the axis on the lower side, then, in the above,

$$SN = SP\cos(\theta - \alpha),$$

so that the equation becomes

$$\frac{l}{r} = 1 + e\cos(\theta - \alpha) \quad\ldots\ldots\ldots\ldots\ldots (2).$$

In fact, the coordinate $\theta$ is now increased by $\alpha$ for each point.

**225. To deduce the form of the curves from the polar equation.**

Being given the polar equation

$$\frac{l}{r} = 1 + e\cos\theta,$$

we can obtain a general idea of the form of the curves with great ease.

We remark, in the first place, that $r$ is infinite only when

$$1 + e\cos\theta = 0, \quad i.e. \text{ when } \cos\theta = -1/e.$$

This gives no real value of $\theta$ if $e$ be less than unity, so that the curve can only proceed to infinity when $e$ is equal to or greater than unity. We infer that the ellipse is a closed curve, and the parabola has an infinite radius vector only when $\cos\theta = -1$, *i.e.* when $\theta = 180°$. The hyperbola has infinite radii vectores in *two* directions—viz., those corresponding to the *two* supplementary values of $\theta$ derivable from $\cos\theta = -1/e$. These directions are equally inclined at an angle $\cos^{-1}(-1/e)$ to the initial line.

We may add that these two infinite radii vectores are the lines through the focus parallel to the asymptotes (for they meet the curve in a point at infinity), and clearly each makes an angle $\sec^{-1}(e)$ with the direction of $XS$ produced. This is, of course, equivalent to saying that the eccentricity is equal to the secant of half the angle between the asymptotes.

We shall now go through the tracing of the hyperbola in detail, leaving the discussion of the ellipse and parabola, which are easier in detail but the same in principle, as an exercise for the reader—an exercise, however, which should by no means be omitted.

The value of $r$ corresponding to any value of $\theta$ is given by
$$r = \frac{l}{1 + e\cos\theta}.$$

When $\theta = 0$, $$r = \frac{l}{1 + e}.$$

Suppose $A$ is the position of this point.

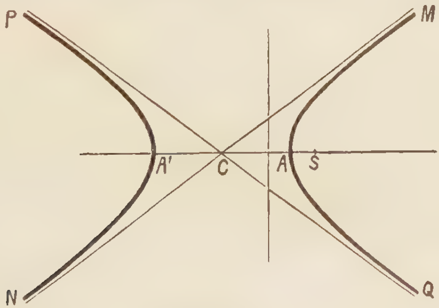



Fig. 80.

As $\theta$ gradually increases, $\cos\theta$ diminishes, so that $r$ increases. This continues to be the case until
$$1 + e\cos\theta$$
becomes zero, when $r$ is infinite. Thus we get the upper portion of the right-hand branch $AM$.

As $\theta$ increases beyond the value which makes $1 + e\cos\theta$

vanish, $1+e\cos\theta$ becomes a very small negative quantity. Hence $r$ suddenly becomes a very large *negative* quantity. That is, the point is now very far away on the *lower* portion of the left-hand branch.

As $\theta$ gradually increases to $180°$, $\cos\theta$ (which is negative) increases in numerical value so that $1+e\cos\theta$, while it decreases in absolute value (being negative), increases numerically, and hence $r$ gradually diminishes in numerical value until $\theta = 180°$. We have now traversed the lower portion $NA'$ of the left-hand branch.

Next, as $\theta$ increases beyond $180°$, $\cos\theta$ diminishes in numerical value again and is negative, so that $1+e\cos\theta$ diminishes and $r$ increases again until it becomes infinite, when $1+e\cos\theta = 0$. For this value of $\theta$, which clearly lies between $180°$ and $270°$, the point is at infinity in the upper portion of the left-hand branch $A'P$.

Finally, when $\theta$ increases beyond this value to $360°$, the tracing point gradually moves from infinity on the lower portion $QA$ of the right-hand branch back again to $A$, which it reaches when $\theta = 360°$. The circuit is thus complete.

*Example.—The latus rectum of a conic is* 6 *and its eccentricity* $\frac{1}{2}$. *Find the length of the focal chord making an angle of* 45° *with the major axis.*

We here use the polar equation.

As the semi-latus rectum is 3 and eccentricity $\frac{1}{2}$, the equation is

$$\frac{3}{r} = 1 + \tfrac{1}{2}\cos\theta.$$

If $PSP'$ be the focal chord, then the value of $\theta$ for $P$ is $PSX = 45°$, and for $P'$ it is the re-entrant angle $ASP' = 225°$.

Thus $\qquad \dfrac{3}{SP} = 1 + \dfrac{1}{2\sqrt{2}}$ and hence $SP = \dfrac{6\sqrt{2}}{2\sqrt{2}+1}$.

Similarly, $\dfrac{3}{SP'} = 1 - \dfrac{1}{2\sqrt{2}}$ and $SP' = \dfrac{6\sqrt{2}}{2\sqrt{2}-1}$.

Finally, the whole chord $= SP + SP'$

$$= 6\sqrt{2}\left\{\frac{1}{2\sqrt{2}+1} + \frac{1}{2\sqrt{2}-1}\right\}$$

$$= 6\sqrt{2}\left\{\frac{2\sqrt{2}-1}{7} + \frac{2\sqrt{2}+1}{7}\right\}$$

$$= \frac{6\sqrt{2}}{7} \times 4\sqrt{2} = \frac{48}{7} = 6\tfrac{6}{7}.$$

**226. The semi-latus rectum is a harmonic mean between the segments of any focal chord.**

We have merely to prove that, if $PSP'$ be any focal chord, then $\frac{2}{l} = \frac{1}{SP} + \frac{1}{SP'}$. (*Tut. Alg.*, II., § 210.)

Now suppose the value of $\theta$ for $P$ (*i.e.* $\angle ASP) = \alpha$; then that for $P'$ is $\pi + \alpha$. Hence, as the polar equation is

$$l/r = 1 + e\cos\theta,$$

$$\frac{l}{SP} = 1 + e\cos\alpha,$$

$$\frac{l}{SP'} = 1 + e\cos(\pi+\alpha) = 1 - e\cos\alpha;$$

and, adding these two equations we have at once

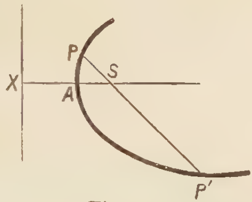



Fig. 81.

$$\frac{l}{SP} + \frac{l}{SP'} = 2$$

or

$$\frac{2}{l} = \frac{1}{SP} + \frac{1}{SP'} \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(3)}.$$

**Exercises.**

1. The equation $1/r = A + B\cos\theta$ denotes a conic for which $l = 1/A$, $e = B/A$.

2. Show that the polar equation of a parabola may be written in the form $r^{\frac{1}{2}}\cos\frac{1}{2}\theta = a^{\frac{1}{2}}$, where $4a$ is the latus rectum.

3. Show that the equation $1/r = A + B\cos\theta + C\sin\theta$ may be always reduced to the form $l/r = 1 + e\cos(\theta - \alpha)$, and hence that it always denotes a conic.

4. Prove that the equations $l/r = 1 + e\cos\theta$ and $l/r = e\cos\theta - 1$ represent the same curve.

5. The latus rectum of a conic is 5 and its eccentricity $\frac{2}{3}$. Find the length of the focal chord making an angle 60° with the major axis.

6. Trace the curves

$3/r = 1 + \frac{1}{2}\cos\theta,\ \ 2/r = 1 + \cos\theta,\ \ 3/r = 1 + 2\cos\theta,\ \ 1/r = 2 + 3\cos\theta.$

7. If $PSP'$ be a focal chord of a conic making an angle $\theta$ with the axis, show that $\frac{1}{SP} + \frac{1}{SP'} = \frac{2}{l}$ and $\frac{1}{SP} - \frac{1}{SP'} = \frac{2e\cos\theta}{l}$.

8. With the notation of the last question, show that

$$PP' = 2l/(1 - e^2\cos^2\theta).$$

227. **To find the equation of the chord joining the two points whose vectorial angles are $(\alpha+\beta)$, $(\alpha-\beta)$ on the curve** $\quad l/r = 1+e\cos\theta$.

[By the vectorial angle is meant the polar coordinate $\theta$ of a point, viz., $\angle ASP$.]

Let $r_1$, $r_2$ be the corresponding radii vectores. Then

$$l/r_1 = 1+e\cos(\alpha+\beta),\quad l/r_2 = 1+e\cos(\alpha-\beta) \;\dots\; (\text{A}).$$

Now the polar equation of the line joining

$$(r_1,\, \alpha+\beta),\ (r_2,\, \alpha-\beta)$$

is

$$r_1 r_2 \sin(\overline{\alpha+\beta}-\overline{\alpha-\beta}) - r_2 r\sin(\theta-\overline{\alpha-\beta}) + r r_1 \sin(\theta-\overline{\alpha+\beta}) = 0. \qquad \text{(Part I., § 10, G)}$$

Multiplying by $l/rr_1r_2$ and substituting from (A), we get

$$\frac{l}{r}\sin 2\beta - (1+e\cos\overline{\alpha+\beta})\sin(\theta-\alpha+\beta) + (1+e\cos\overline{\alpha-\beta})\sin(\theta-\alpha-\beta) = 0.$$

Now

$$-\sin(\theta-\alpha+\beta)+\sin(\theta-\alpha-\beta) = -2\cos(\theta-\alpha)\sin\beta,$$

and $\quad e(\cos\overline{\alpha-\beta}\sin\overline{\theta-\alpha-\beta} - \cos\overline{\alpha+\beta}\sin\overline{\theta-\alpha+\beta})$

$$= \tfrac{1}{2}e\left\{\sin(\theta-2\beta)+\sin(\theta-2\alpha)-\sin(\theta+2\beta)-\sin(\theta-2\alpha)\right\}$$

$$= \tfrac{1}{2}e\left\{\sin(\theta-2\beta)-\sin(\theta+2\beta)\right\} = -e\cos\theta\sin 2\beta.$$

Substituting in the equation of the straight line, we get

$$\frac{l}{r}\sin 2\beta - 2\cos(\theta-\alpha)\sin\beta - e\cos\theta\sin 2\beta = 0$$

or

$$\frac{l}{r} = e\cos\theta + \sec\beta\cos(\theta-\alpha) \;\dots\dots\dots\; \textbf{(4)}.$$

**228. Alternative proof.**—The preceding analysis is perhaps just a little tedious, and it may be replaced by shorter methods. Its advantage perhaps lies in the fact that it is perfectly straightforward, for we assume the equation of the line in the most natural form, and then solve the resulting linear equations in the ordinary way.

The form of the equation found shows that the work

would be shortened if we started by assuming the line in the form $$\frac{l}{r} = C\cos(\theta-\alpha) + D\cos\theta.*$$

Then this equation and $l/r = 1 + e\cos\theta$ must give the same value of $r$ when $\theta = \alpha+\beta$, and also when $\theta = \alpha-\beta$.

Hence $1 + e\cos(\alpha+\beta) = C\cos\beta + D\cos(\alpha+\beta)$

and $1 + e\cos(\alpha-\beta) = C\cos\beta + D\cos(\alpha-\beta)$.

On subtraction we find at once

$$D = e,$$

and then on substitution $C = \sec\beta$ immediately.

Hence the equation is

$$\frac{l}{r} = e\cos\theta + \sec\beta\cos(\theta-\alpha).$$

**229. Equation of the tangent at the point $\alpha$.**

If, in the equation of the chord joining $\alpha+\beta$ to $\alpha-\beta$, we make $\beta$ smaller and smaller, and ultimately zero, we obtain the tangent at the point $\alpha$.

Its equation is therefore found by making $\beta = 0$ in

$$\frac{l}{r} = e\cos\theta + \sec\beta\cos(\theta-\alpha),$$

*i.e.* it is $$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha) \quad \ldots\ldots\ldots\ldots (5).$$

**Caution.**—When the equation of the tangent is to be found, the main portions of § 227 or § 228 must first be given.

230. **Alternative method of finding the polar equation to the tangent.**

The following method will be found instructive.

The equation of the tangent can be easily obtained by changing into Cartesian coordinates.

In fact the curve is

$$\frac{l}{r} = 1 + e\cos\theta \quad \text{or} \quad r = l - er\cos\theta.$$

Therefore, squaring, we have

$$x^2 + y^2 = (l - ex)^2,$$

the equation of the conic.

---

* This *does* represent a straight line, for on multiplying up and changing into Cartesians it is of the first degree.

This equation may be written

$$x^2(1-e^2)+y^2+2lex-l^2=0,$$

and hence the equation of the tangent at $(x_1, y_1)$ is

$$xx'(1-e^2)+yy'+le(x+x')-l^2=0$$

or $$xx'+yy'=l^2-le(x+x')+e^2xx'=(l-ex)(l-ex').$$

Now suppose the polar coordinates of $(x', y')$ are $r$, $\alpha$, and change back into polars.

Then the above becomes

$$rr_1(\cos\theta\cos\alpha+\sin\theta\sin\alpha)=(l-er\cos\theta)r_1,$$

for $r_1=l-ex'$, since $(x', y')$ is on the curve;

$$\therefore\quad r\cos(\theta-\alpha)=l-er\cos\theta;$$

$$\therefore\quad l=r\left\{e\cos\theta+\cos(\theta-\alpha)\right\};$$

leading, as before, to $\dfrac{l}{r}=e\cos\theta+\cos(\theta-\alpha)$.

231. **Geometrical interpretation of the equation of the tangent.**

Suppose $P$ is the point $\alpha$, and that $T$ is any point $r$, $\theta$ on the tangent at $P$. Draw $TM$ perpendicular to $SP$, and $TN$ perpendicular to the directrix.

Then, since

$$\angle ASP=\alpha \quad\text{and}\quad \angle AST=\theta,$$
$$TSP=\alpha-\theta.$$

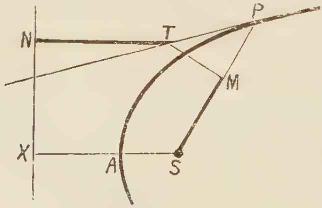



Fig. 82.

Now, by the equation of the tangent

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha)$$

or $$l-er\cos\theta=r\cos(\theta-\alpha)=ST\cos TSP=SM,$$

and $l-er\cos\theta=e.SX-er\cos\theta=e(SX-r\cos\theta)=e.TN.$

Hence $$SM=eTN;$$

$$\therefore\quad SM:TN=SA:AX.$$

Thus the geometrical interpretation of the polar equation of the tangent leads at once to what is known as *Adam's property of the conic.*

**Exercises.**

9. Show that the equation of the tangent at the point $\alpha$ when transformed into Cartesian coordinates is

$$x(e+\cos\alpha)+y\sin\alpha = l.$$

10. Show that the equation of the perpendicular from $S$ on the tangent at the point $\alpha$ is

$$x\sin\alpha - y(e+\cos\alpha) = 0.$$

11. Prove that the locus of the foot of the perpendicular from $S$ on the tangents of the conic is obtained by eliminating $\alpha$ between the equations of the last two examples, and hence show that the equation of the locus is $$(l-ex)^2+e^2y^2 = x^2+y^2.$$

What does this equation represent?

**Illustrative Examples.**

(i.) **The tangents drawn from any point to a conic subtend equal angles at the focus.**

Suppose the tangents at $P$ and $Q$ meet in $T$; then we have to show that $TS$ bisects the angle $PSQ$.

Let the points $P$ and $Q$ have vectorial angles $\alpha$ and $\beta$; then the equations of the tangents $TP$, $TQ$ are

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha)$$

and $$\frac{l}{r} = e\cos\theta + \cos(\theta-\beta).$$

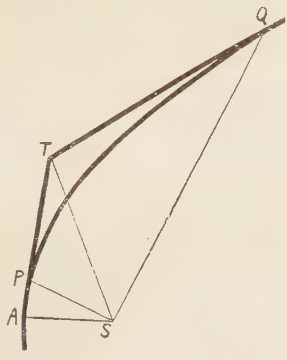



Fig 83.

To find the coordinates of $T$, we have to solve these equations for $r$ and $\theta$. Subtracting, we obtain at once $\cos(\theta-\alpha) = \cos(\theta-\beta)$.

Now the angles cannot be equal, for then $\alpha=\beta$. Hence

$$\theta-\alpha = -(\theta-\beta) \quad \text{or} \quad \theta-\alpha = \beta-\theta.$$

Thus

$$\angle AST - \angle ASP = \angle ASQ - \angle AST,$$

*i.e.* $$\angle TSP = \angle TSQ,$$

which shows at once that $TS$ bisects the angle $PSQ$

(ii.) **The portion of a tangent subtended between the point of contact and the directrix subtends a right angle at the focus.**

Suppose the tangent at $P$ meets the directrix in $T$; then we have to show that $PST$ is a right angle.

For any point on the directrix we have

$$r\cos\theta = SX = \frac{l}{e} \text{ or } \frac{l}{r} = e\cos\theta,$$

for $\qquad l = e.SX. \qquad$ (by § 62)

Now the tangent at the point $\alpha$ is

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha),$$

and, where this meets the directrix,

$$\frac{l}{r} = e\cos\theta.$$

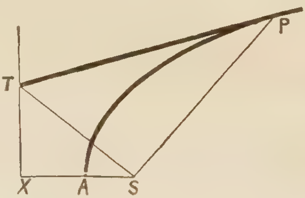


Fig. 84.

To find the coordinates of $T$, we have to solve these two equations for $r$ and $\theta$. Subtracting, we have

$$\cos(\theta-\alpha) = 0;$$

$$\therefore \quad \theta-\alpha = \pm\frac{\pi}{2},$$

which shows that

$$\angle PST = \angle PSA - \angle TSA = \theta-\alpha = \pm\frac{\pi}{2},$$

and hence proves the result required.

## MISCELLANEOUS EXERCISES ON CHAP XVII.

12. Show that, if $PP'$ and $QQ'$ be two focal chords of a parabola at right angles, then $\qquad \dfrac{1}{PP'} + \dfrac{1}{QQ'} = \dfrac{1}{2l}.$

13. If the conic $l/r = 1 + e\cos\theta$ be an ellipse, and $a$, $b$ its minor axes, show that $\quad a = l/(1-e^2)$ and $b = l/\sqrt{1-e^2}$.

14. Find the corresponding results for the hyperbola.

15. Show that the polar equation of the auxiliary circle of the conic $l/r = 1 + e\cos\theta$ is $\quad r^2(1-e^2) - 2ler\cos\theta + l^2 = 0$.

16. The general equation of all conics having the same focus and directrix is $1/r = a + p\cos\theta$, where $p$ is the same for all conics of the system.

17. Find three focal radii vectores of an ellipse such that their lengths shall be in harmonic progression while their angular coordinates are in arithmetical progression.

18. If, with the focus of a parabola as centre, a circle be described passing through the vertex, the rectangle under the intercepts of any focal chord between the circle and parabola is constant.

19. Find the polar coordinates of the point of intersection of the tangents at the points $\alpha$ and $\beta$, and hence prove that, if tangents at $P$ and $Q$ to a parabola meet in $T$, then $ST^2 = SP \cdot SQ$.

20. Find the condition that the line whose equation is

$$1/r = A\cos\theta + B\sin\theta$$

should touch the conic whose equation is $l/r = 1 + e\cos\theta$.

21. Two conics have the same focus and directrix. If any tangent to one cut the other in $P$ and $Q$, $\angle PSQ$ is constant, and $\cos\frac{1}{2}PSQ = e/e'$ ($e$, $e'$ being the eccentricities).

22. A system of conics have the same focus and latus rectum. Prove that the tangents at all points on a fixed line through focus cut the latus rectum produced at the same distance from focus.

23. $S$ is the common focus of two parabolas whose vertices are at $A$ and $A'$. If $A$ be the mid-point of the straight line $SA'$, and $Q'QSPP'$ be a focal chord cutting the parabolas in the points $Q'$, $Q$, $P$, $P'$, show that $SQ'$ is bisected at $Q$, and that tangents to the parabolas at $Q'$ and $P$ respectively are at right angles, and find the locus of their intersection.

24. Prove that the locus of the intersections of tangents at the extremities of perpendicular focal radii of a conic is another conic having the same focus.

## EXAMINATION PAPER V

1 State (without proof) the chief properties of pole and polar with respect to a conic When are two lines said to be *conjugate*?

Prove that the pole of any line parallel to an asymptote is on that asymptote.

2. Find the equation of the polar of $(x', y')$ with respect to the hyperbola $xy = c^2$.

3. Find the pole of $3x-2y = 1$ with respect to the conic

$$9x^2-8y^2-16x+8y+6 = 0.$$

4. Explain how you would obtain geometrically the polar of a point within an ellipse with respect to that ellipse.

5. A point $P$ is such that the line drawn through it perpendicular to its polar with respect to the parabola $y^2 = 4ax$ touches the parabola $x^2 = 4by$. Show that $P$ lies on the line $2ax+by+4a^2 = 0$.

6. Show that, if two tangents to a parabola make complementary angles with the axis, their chord of contact will cut the axis in a fixed point.

7. Straight lines are drawn cutting the ellipse $x^2/a^2+y^2/b^2 = 1$ so that the sum of the eccentric angles of the two points in which any of the straight lines cuts the ellipse is constant and equal to $2\epsilon$. Prove that the locus of the poles of these straight lines is the straight line

$$ay = bx \tan \epsilon.$$

8. Find the equation of the normal to an ellipse in terms of the eccentric angle.

Show that four normals can be drawn to an ellipse through a given point, and find an equation for the inclination of the normal to the axis.

9. Prove that the area of the ellipse $x^2/a^2+y^2/b^2 = 1$ is $\pi ab$.

10. Find the polar equation of a tangent to a parabola $y^2 = 4ax$, the focus being the pole, and reduce it to the form

$$r = a \sec\frac{\alpha}{2} \sec\left(\theta-\frac{\alpha}{2}\right).$$

Prove that the tangents at the ends of a focal chord of a parabola meet at right angles on the directrix.

# CHAPTER XVIII.

## SYSTEMS OF CONICS.

**232. Abbreviated notation.**

In this chapter we shall explain some very important principles connected with the equations of conics.

We shall frequently use the equation in the abbreviated form

$$S = 0$$

where $S$ means

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

and we may write $S'$ for the same equation in which the coefficients have dashes, *i.e.* for

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c'.$$

In like manner we shall use a single letter to denote the equation of a straight line, say

$$u = 0,$$

where $u$ is, of course, an expression of the first degree in $x$ and $y$.

Thus we write

$$u \equiv lx + my + n = 0$$

for the equation.

**233. If $S=0$ and $S'=0$ be the equations of two conics, then $S+kS'=0$ for all constant values of $k$ is the equation of a conic through the points of intersection of $S$ and $S'$.**

For, in the first place, the equation

$$S+kS'=0$$

is of the second degree since $S$ and $S'$ are, and therefore it denotes a conic.

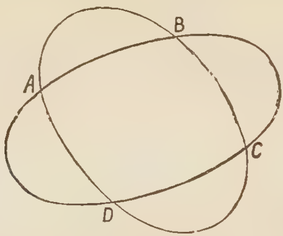



Fig. 85.

Next, if $A$ be a point of intersection of $S=0$ and $S'=0$, then the coordinates of $A$ satisfy

$$S=0 \quad \text{and} \quad S'=0.$$

Therefore its coordinates satisfy

$$S+kS'=0,$$

*i.e.* the conic $S+kS'=0$ passes through $A$.

Similarly, it passes through the other points of intersection of $S=0$ and $S'=0$.

Thus $S+kS'=0$ represents a conic passing through *all* the points of intersection of $S=0$ and $S'=0$.

The reader will notice the similarity between this reasoning and that in § 31.

**Exercises.**

1. If $S=0$ and $S'=0$ be the equations of two circles, the equation $S+\lambda S'=0$ denotes a circle through their points of intersection.

2. If $S=0$ and $S'=0$ be both rectangular hyperbolas, show that every conic of the system $S+\lambda S'=0$ is a rectangular hyperbola. Deduce that all conics through the points of intersection of two rectangular hyperbolas are rectangular hyperbolas.

**234. Two conics intersect in four points, real or imaginary.**

We did not make any assumption in the preceding article as to the number of points in which two curves of the second degree intersect. We shall now show that "two curves of the second degree intersect in four points, real or imaginary."

Suppose, in fact, that their equations are

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$
$$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0;$$

then, to find their points of intersection, we have to solve these equations for $x$ and $y$. To do this we arrange them both as quadratics in $x$, then eliminate $x$ and obtain an equation for $y$, which is of the fourth degree and therefore has four roots, real or imaginary.

Thus we have

$$\left.\begin{aligned} x^2.a+2x(hy+g)+by^2+2fy+c&=0\\ x^2.a'+2x(h'y+g')+b'y^2+2f'y+c'&=0\end{aligned}\right\}\quad\ldots\ (\text{A}).$$

Eliminating in the usual way, we have

$$\{a.2(h'y+g')-a'.2(hy+g)\}$$
$$\{2(hy+g)(b'y^2+2f'y+c')-2(h'y+g')(by^2+2fy+c)\}$$
$$=\{a(b'y^2+2f'y+c')-a'(by^2+2fy+c)\}^2,$$

and this is clearly of the fourth degree in $y$. Hence there are four roots, say $y_1, y_2, y_3, y_4$ (*Tut. Alg.*, II, § 371). Since we can eliminate $x^2$ from equation (A), and obtain a *linear* equation for $x$ in terms of $y$, it is clear that to each value of $y$ there is only one value of $x$.

Thus, if $x_1, x_2, x_3, x_4$ be the four corresponding values of $x$, we see that there are four points of intersection, viz., $(x_1, y_1)$, $(x_2\, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$.

Note.—Of course, some of these four points may be imaginary. It is easy to see (*Tut. Alg.*, § 378) that they become imaginary in pairs.

**Exercise.**

3. Without using the above analysis, prove the truth of the above proposition when one of the conics is a pair of straight lines.

235. **Any conic passing through the four points of intersection of two given conics may be made to satisfy one, and only one, *other* condition.**

If $S=0$ and $S'=0$ be the equations of two conics, the equation $S+kS'=0$ denotes a conic through their points of intersection. Since this equation contains the arbitrary constant $k$, we can make the conic satisfy one further condition, the value of $k$ being then determined.

236. **Case in which $S'=0$ is a pair of straight lines.**

If $S'=0$ break up into two factors, say $u.v=0$, then it denotes two straight lines. Each of these lines meets $S=0$ in two points, say $A$, $B$ and $C$, $D$, and then the equation
$$S+kuv=0$$

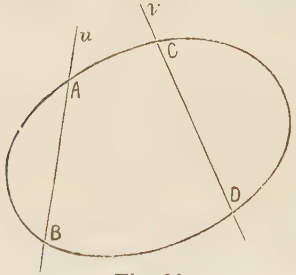



Fig. 86A.

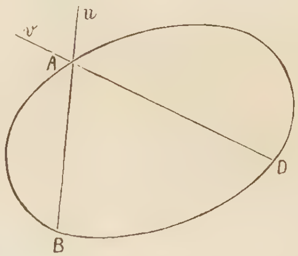



Fig. 86B.

denotes a conic passing through the four points $A$, $B$, $C$, $D$.

Suppose, now, that $D$ remains fixed, and the line $v$ turns round so that $C$ approaches and ultimately coincides with $A$; then the conic $S+kuv=0$ meets $S=0$ in two coincident points at $A$, *i.e.* touches it.

Hence, if the straight lines $u=0$ and $v=0$ meet on the conic $S=0$, then the conic $S+kuv=0$ touches $S=0$ in the point of intersection of $u=0$ and $v=0$.

*Example* (i.). *Find the equation of a conic through the points of intersection of the conics* $x^2+xy+y^2+3y=0$ *and* $2xy+3y+4=0$, *and also through the point* $(1, 1)$.

The equation of any conic passing through the points of intersection of the given conics is of the form
$$x^2+xy+y^2+3y+k(2xy+3y+4)=0,$$

If it passes through the point $(1, 1)$, we must have

$$6 + 9k = 0 \quad \text{or} \quad k = -\tfrac{2}{3}.$$

Thus the required equation is

$$3(x^2 + xy + y^2 + 3y) - 2(2xy + 3y + 4) = 0$$

or

$$3x^2 - xy + 3y^2 + 3y - 8 = 0.$$

*Example* (ii.). *Find the equation of a conic through the points where the axes meet the conic* $3x^2 + 4xy + y^2 - y + 2 = 0$, *and through the point* $(-1, 1)$.

The equation is of the form

$$(3x^2 + 4xy + y^2 - y + 2) + kxy = 0,$$

where $k$ has to be determined by the condition that the required conic passes through $(-1, 1)$.

Hence

$$1 - k = 0 \quad \text{or} \quad k = 1.$$

Thus the equation is $3x^2 + 5xy + y^2 - y + 2 = 0$.

**Exercises.**

4. Find the equation of a conic through the points of intersection of $3x^2 + y^2 = 4$ and $xy = 5$, and through the point $(-1, 2)$.

5. Find the equation of a conic through the points of intersection of

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

and

$$S' \equiv a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0,$$

and through the origin.

6. Find the equation of a conic through the points of intersection of $x^2 + xy + y^2 = 3$ and $2x^2 + xy - y^2 + 3x = 0$ and having an asymptote parallel to the axis of $x$.

[The terms of the second degree must contain $y$ as a factor (see § 110).]

7. Show that two parabolas can be drawn through the four points of intersection of two conics.

[Here the terms of the second degree must be a perfect square in the resulting equation.]

### 237. Interpretation of $S + ku^2 = 0$.

Next suppose $C$ and $D$ both move round the conic so that $C$ approaches $A$ and ultimately coincides with it, while $D$ approaches $B$ and ultimately coincides with it.

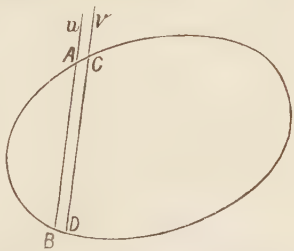


Fig. 86c.

Then the conic $S + kuv = 0$ meets $S = 0$ in two ultimately coincident points at $A$, and in two ultimately coincident

points at $B$. Also under these circumstances $v=0$ is identically the same as $u=0$, and we thus see that

**The conic $S+ku^2=0$ touches the conic $S=0$ in the two points in which it is met by the straight line $u=0$.**

Conics which touch in two points in this manner are said to have **double contact.**

238. **Conics touching the axes of coordinates.**

The general equation of a conic touching the conic $S=0$ where it is met by the line $u=0$ is $S+\lambda u^2=0$.

Now let $S=0$ represent the axes, whose equation is $xy=0$.

Then we see that the general equation of a conic touching the axes when they are met by the line $lx+my-1=0$ is

$$xy+\lambda(lx+my-1)^2=0,$$

or as we may write it, putting $1/\lambda=2\mu$,

$$(lx+my-1)^2+2\mu xy=0 \quad\text{.....................}\quad (1).$$

239. **Parabola touching the axes of coordinates.**

The equation of a conic touching the axes of coordinates is of the form
$$(lx+my-1)^2+2\mu xy=0.$$

For this to be a parabola the terms of the second degree must form a perfect square.

These terms are $\quad x^2.l^2+2xy(lm+\mu)+y^2.m^2,$

so that $\quad l^2m^2=(lm+\mu)^2 \quad$ or $\quad \mu=-2lm.$

[The root $\mu=0$ is inadmissible, as it leads merely to two coincident straight lines $lx+my=1$.]

Thus, since there is only one value of $\mu$, there is only one parabola touching the axes in the two points, and its equation is

$$(lx+my-1)^2-4lmxy=0 \quad\text{or}\quad (lx-my)^2-2lx-2my-1=0.$$

This equation can be put into a remarkable form.

In fact, since

$$(lx+my-1)^2=4lmxy,$$

we have $\quad lx+my-1=\pm 2\sqrt{lmxy},$

*i.e.* $\quad lx+my\pm 2\sqrt{lmxy}=1.$

Hence, equating the square roots of the two sides, we get

$$\sqrt{lx}+\sqrt{my}=1 \quad\text{...........}\quad (2),$$

where either sign may be taken with each square root.

NOTE.—The parabola touches the axes where the line $lx+my-1=0$ meets them.

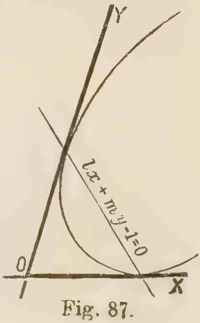



Fig. 87.

**240. To find the equation of the tangents drawn from the point $T(x', y')$ to the conic**

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

If $P$, $Q$ be the points of contact, then the tangents $TP$, $TQ$ form a conic having double contact with the given one at $P$ and $Q$. But, $PQ$ being the polar of $T$, its equation is $\quad x(ax'+hy'+g)+y(hx'+by'+f)+gx'+fy'+c=0.$

Hence, by § 237, the required equation of the tangents is of the form

$$ax^2+2hxy+by^2+2gx+2fy+c$$
$$+\lambda\{x(ax'+hy'+g)+y(hx'+by+f)+gx'+fy'+c\}^2=0.$$

The value of $\lambda$ is determined by the consideration that if this equation represents the tangents from $T$ which both pass through $T$ it must be satisfied by $x'$, $y'$, the coordinates of $T$.

$$\therefore\ ax'^2+2hx'y'+by'^2+2gx'+2fy'+c$$
$$+\lambda\{ax'^2+2hx'y'+by'^2+2gx'+2fy'+c\}^2=0.$$

Thus $\quad \lambda=-\dfrac{1}{ax'^2+2hx'y'+by'^2+2gx'+2fy'+c}$

and the required equation is

$$(ax^2+2hxy+by^2+2gx+2fy+c)$$
$$(ax'^2+2hxy'+by'^2+2gx'+2fy'+c)$$
$$=\{axx'+h(xy'+x'y)+byy'+g(x+x')+f(y+y')+c\}^2,$$

as we have already found (§ 138).

**Exercises.**

8. Write down the equation of the tangents from the point (1, 1) to the conic $x^2+xy+y^2=5$.

Find from first principles by the methods of § 240 the equation of the tangents from the point $(x', y')$ to the following conics, and verify that the results agree with those obtained by taking particular cases of the general result.

9. $x^2+y^2+2gx+2fy+c=0.$ 10. $y^2-4ax=0.$

11. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0.$ 12. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-1=0.$

13. Write down the conditions that the tangents in Exx. 9-12 are at right angles, and hence deduce the equations of the director circles. What does the director circle become in the case of the parabola?

14. Find the equation of a conic passing through the point $(-1, 1)$ and having double contact with $3x^2 - xy + 2y^2 + x + y + 1 = 0$ where it meets the axis of $x$.

15. Find the equation of a conic passing through the origin and through the points in which the straight lines $x + y + 1 = 0$, $x - 3y + 2 = 0$ meet the conic $2x^2 + 3y^2 = 6$.

16. Show that only one parabola can be drawn touching a given conic in two given points.

[Take $lx + my + n = 0$ as the line joining the two points and find $k$, so that the resulting conic may be a parabola.]

17. Show that only one rectangular hyperbola can be drawn touching a given conic in two given points.

**18. Show that the equation of a parabola touching the axes in points $A$, $B$, distant $a$, $b$ from the origin may be reduced to the form** $\pm\sqrt{\dfrac{x}{a}} \pm \sqrt{\dfrac{y}{b}} = 1$.

19. Further prove that (i.) for the part of the parabola in Ex. 18 within the triangle $OAB$ both positive signs must be taken; (ii.) for the part beyond the point $A$ the signs are $+$, $-$, and for the part beyond $B$ the signs are $-$, $+$.

20. Find the equation of a parabola referred to the tangents at the ends of the latus rectum as axes.

[Note that they are at right angles.]

**21. Show that for all values of $A_1$, $A_2$, $A_3$ the equation**

$$\frac{A_1}{x\cos\alpha_1 + y\sin\alpha_1 - p_1} + \frac{A_2}{x\cos\alpha_2 + y\sin\alpha_2 - p_2} + \frac{A_3}{x\cos\alpha_3 + y\sin\alpha_3 - p_3} = 0$$

**represents a conic circumscribing the triangle formed by the three lines** $x\cos\alpha_1 + y\sin\alpha_1 - p_1 = 0$, **&c.**

[Note that it is of the second degree, and that the point of intersection of any two of the lines is on it.]

22. Find the conditions in Ex. 21 that the equation may represent a circle, and show that they may be written in the form

$$A_1\cos(\alpha_2 + \alpha_3) + A_2\cos(\alpha_3 + \alpha_1) + A_3\cos(\alpha_1 + \alpha_2) = 0,$$
$$A_1\sin(\alpha_2 + \alpha_3) + A_2\sin(\alpha_3 + \alpha_1) + A_3\sin(\alpha_1 + \alpha_2) = 0.$$

[Equate the coefficients of $x^2$ and $y^2$ and the coefficient of $xy$ and zero, the axes being supposed rectangular.]

23. Hence, by solving the equations for $A_1$, $A_2$, $A_3$, show that **the equation of the circumcircle may be written**

$$\frac{\sin(\alpha_2 - \alpha_3)}{x\cos\alpha_1 + y\sin\alpha_1 - p_1} + \frac{\sin(\alpha_3 - \alpha_1)}{x\cos\alpha_2 + y\sin\alpha_2 - p_2} + \frac{\sin(\alpha_1 - \alpha_2)}{x\cos\alpha_3 + y\sin\alpha_3 - p_3} = 0.$$

241. **One conic, and one only, can be drawn through five given points.**

We have seen that we can draw a conic through the four points of intersection of two conics, and through any other point in the plane. Thus we have obtained a conic passing through the five points. This can always be done.

If $A, B, C, D, E$ be the points, then through the points $A, B, C, D$ we have a system of conics, of which the pair of lines $AB, CD$ is one, and the pair of lines $BC, AD$ another.

Any conic through their common points ($A$, $B$, $C$, $D$) is $S+kS'=0$, where

$S=0$ is the equation of $AB$ and $CD$,

$S'=0$ is the equation of $BC$ and $AD$.

Finally, we can find $k$ such that the conic

$$S+kS'=0$$

shall pass through $E$. The equation for $k$ being of the first degree, only one such conic can be drawn.

The above result can also be obtained thus :—

The equation of a conic is always of the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Since we can divide across by any coefficient, the equation contains only five constants actually, although it apparently contains six.

Hence, since we have five constants at our disposal, *a conic may be made to satisfy five conditions* in just the same manner as a straight line may be made to satisfy two.

If five points are given, then, on substituting their coordinates in $ax^2+2hxy+by^2+2gx+2fy+c=0$,

we obtain *five simultaneous equations of the first degree* for the ratios of the coefficients, and these are sufficient to determine them.

In general it is quite clear that when a conic satisfies any condition the coefficients in its equation must satisfy a certain relation.

In actual practice the method of last article is always the more satisfactory one to use.

242. **Conics through four given points.**

Let the points be $A$, $B$, $C$, $D$. Join $AB$, $CD$, and produce them to meet in $O$. Then take the lines $OAB$, $OCD$ for axes of $x$ and $y$ respectively.

Let $OA = \alpha,\ OB = \beta,\ OC = \gamma,\ OD = \delta.$

Then the equation of $AC$ is

$$\frac{x}{\alpha} + \frac{y}{\gamma} = 1,$$

and that of $BD$ is

$$\frac{x}{\beta} + \frac{y}{\delta} = 1.$$

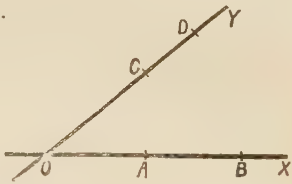



Fig. 88.

Also $AB$ is $y = 0$

and $CD$ is $x = 0.$

Hence two conics through the four points are

$$xy = 0 \quad \text{and} \quad \left(\frac{x}{\alpha} + \frac{y}{\gamma} - 1\right)\left(\frac{x}{\beta} + \frac{y}{\delta} - 1\right) = 0.$$

Consequently, any other is of the form

$$\left(\frac{x}{\alpha} + \frac{y}{\gamma} - 1\right)\left(\frac{x}{\beta} + \frac{y}{\delta} - 1\right) + \lambda xy = 0 \quad \ldots \ (3).$$

**Exercises.**

24. Find the equation of the conic passing through the points

$(0, 0),\ (1, 0),\ (0, 2),\ (2, 1),\ (4, 2).$

25. Find the equation of the conic passing through the points

$(0, 0),\ (1, 0),\ (0, 2),\ (2, 1),\ (3, 2).$

26. One of the other conics is clearly the line pair

$$\left(\frac{x}{\alpha} + \frac{y}{\delta} - 1\right)\left(\frac{x}{\beta} + \frac{y}{\gamma} - 1\right) = 0.$$

Find the value of $\lambda$, which gives this conic.

243. **Confocal conics.**

We shall now give a short account of the properties of a system of conics having the same foci. Such conics are called **confocal conics.** Many of their properties can be deduced most easily from purely geometrical considerations, but we shall here confine our attention mainly to analytical processes.

**244. Equation of a confocal system.—The general equation of a conic having the same foci as**

$$x^2/a^2+y^2/b^2=1$$

**is**

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

**where λ is a constant.**

In the first place, any conic having the same foci $S$ and $S'$ as the conic $x^2/a^2+y^2/b^2=1$ must have the same centre, and its axes along the same lines, for $SS'$ is the direction of the major axis, and a line perpendicular to it through its middle point the minor axis. Thus these are the same for all the conics.

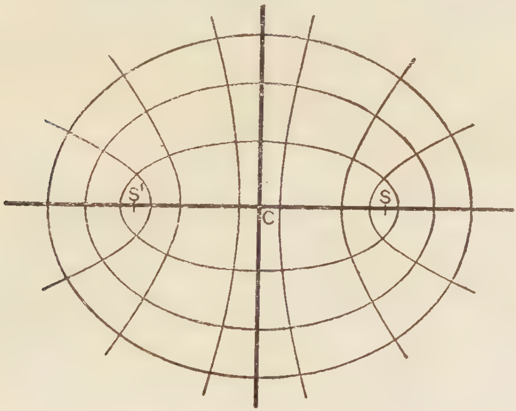



Fig. 89.

Hence the equation of any other conic is of the form

$$x^2/a'^2+y^2/b'^2=1.$$

But $$CS^2=a^2e^2=a^2-b^2; \qquad (\S\ 61)$$

$$\therefore\quad a^2-b^2=a'^2-b'^2;$$

so that, if $a'^2=a^2+\lambda$, we have

$$b'^2=b^2+\lambda;$$

and the equation is $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$ .................. (4).

245. The reader can easily verify the following statements:—

If $a$ be greater than $b$, then the conic

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

(1) is an ellipse for all positive values of $\lambda$ and for such negative values of $\lambda$ as are less than $b^2$ numerically, *i.e.* when $\lambda > -b^2$;

(2) is the line $y=0$, *i.e.* the axis of $x$, when $\lambda=-b^2$;

(3) is an hyperbola when $\lambda < -b^2$ and $> -a^2$;

(4) is the line $x=0$, *i.e.* the axis of $y$, when $\lambda=-a^2$;

(5) is an imaginary ellipse when $\lambda < -a^2$.

**246. Through any point in the plane two conics of the confocal system can be drawn.**

Let $(x_1, y_1)$ be the point. Then we have to determine $\lambda$ so that the conic $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$

may pass through the point $(x_1, y_1)$.

Hence we have $\dfrac{x_1^{\,2}}{a^2+\lambda}+\dfrac{y_1^{\,2}}{b^2+\lambda}=1$

or $$x_1^{\,2}(b^2+\lambda)+y_1^{\,2}(a^2+\lambda)=(a^2+\lambda)(b^2+\lambda),$$

*i.e.* $$\lambda^2+\lambda(a^2+b^2-x_1^{\,2}-y_1^{\,2})+a^2b^2-a^2y_1^{\,2}-b^2x_1^{\,2}=0.$$

This is a quadratic for $\lambda$, and therefore there are two values of $\lambda$ such that the corresponding confocal passes through $(x_1, y_1)$.

*Example.—Find the conics confocal with* $\dfrac{x^2}{2}+\dfrac{y^2}{1}=1$ *which pass through the point* (1, 1).

Here the equation for $\lambda$ is

$$\frac{x_1^{\,2}}{2+\lambda}+\frac{y_1^{\,2}}{1+\lambda}=1 \quad \text{or} \quad \frac{1}{2+\lambda}+\frac{1}{1+\lambda}=1,$$

*i.e.* $$\lambda^2+\lambda-1=0; \quad \text{or} \quad \lambda=\tfrac{1}{2}(-1\pm\sqrt{5}).$$

Thus the conics required are

$$\frac{x^2}{2-\frac{1}{2}(1+\sqrt{5})}+\frac{y^2}{1-\frac{1}{2}(1+\sqrt{5})}=1$$

and $$\frac{x^2}{2-\frac{1}{2}(1-\sqrt{5})}+\frac{y^2}{1-\frac{1}{2}(1-\sqrt{5})}=1,$$

or $$\frac{2x^2}{3-\sqrt{5}}-\frac{2y^2}{\sqrt{5}-1}=1 \quad \text{and} \quad \frac{2x^2}{\sqrt{5}+3}+\frac{2y^2}{\sqrt{5}+1}=1.$$

Clearly the first of these is a hyperbola and the second an ellipse.

**247. Of the two confocals through a given point, one is always an ellipse and the other a hyperbola.**

Suppose $\beta = b^2+\lambda$. Then, when $\beta$ is negative, the conic
$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$
is clearly a hyperbola. When $\beta$ is positive it is an ellipse.

The equation may be written
$$\frac{x^2}{a^2-b^2+\beta}+\frac{y^2}{\beta}=1;$$
and hence, if the conic passes through $(x_1, y_1)$, $\beta$ is given by
$$\frac{x_1^2}{a^2-b^2+\beta}+\frac{y_1^2}{\beta}=1$$
or
$$\beta(a^2-b^2+\beta)=\beta x_1^2+(a^2-b^2+\beta)y_1^2,$$
*i.e.*
$$\beta^2+\beta(a^2-b^2-x_1^2-y_1^2)-(a^2-b^2)y_1^2=0.$$

Now $a > b$, and therefore the product of the roots is negative (*Tut. Alg.*, II., § 156).

Hence the roots are real, one being positive and the other negative. Thus one of the conics is an ellipse and the other a hyperbola.

**248. Two confocal conics intersect each other at right angles.**

Suppose the conics are
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$
$(x_1, y_1)$ being a point of intersection. Then we have to show that the tangents at $(x_1, y_1)$ to the two conics are at right angles.

Now the two tangents are
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \text{ and } \frac{xx_1}{a^2+\lambda}+\frac{yy_1}{b^2+\lambda}=1.$$

These are at right angles if only
$$\frac{x_1^2}{a^2(a^2+\lambda)}+\frac{y_1^2}{b^2(b^2+\lambda)}=0. \quad \text{(Pt. I., § 19.)}$$

Now we have

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1 \text{ and } \frac{x_1^2}{a^2+\lambda}+\frac{y_1^2}{b^2+\lambda}=1.$$

Subtracting these equations, we have

$$x_1^2\left(\frac{1}{a^2}-\frac{1}{a^2+\lambda}\right)+y_1^2\left(\frac{1}{b^2}-\frac{1}{b^2+\lambda}\right)=0$$

or

$$\frac{x_1^2\lambda}{a^2(a^2+\lambda)}+\frac{y_1^2\lambda}{b^2(b^2+\lambda)}=0.$$

$$\therefore \quad \frac{x_1^2}{a^2(a^2+\lambda)}+\frac{y_1^2}{b^2(b^2+\lambda)}=0.$$

But this is precisely the condition that the tangents should be at right angles, and hence the conics cut at right angles at all their points of intersection.

249. **Only one confocal of a given system can be drawn to touch a given straight line.**

Let the equation of the straight line be

$$lx+my=1.$$

The condition that this should touch the conic

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

is

$$(a^2+\lambda)l^2+(b^2+\lambda)m^2=1. \qquad (\S\ 141)$$

Thus there is only one confocal touching the given line, for this equation is linear in $\lambda$, and therefore has only one root.

### Exercises.

27. Of the conics through the point (1, 2) confocal with $2x^2+y^2=1$, show that one is an ellipse and the other a hyperbola.

28. From the property $SP+S'P=$ constant show that only one ellipse having given foci can be drawn through a given point.

Prove the same property for a hyperbola.

29. Deduce the result of § 248 from the fact that the tangent at any point to a central conic bisects the internal or external angle between the focal distances.

30. Show that the conic confocal to $x^2/2+y^2=1$ which touches $x+y=6$ is $2x^2/37+2y^2/35=1$.

31. Find the equation to the hyperbola which passes through the point $(a\cos\alpha, b\sin\alpha)$ and is confocal with the ellipse $x^2/a^2+y^2/b^2=1$.

250. **To find the locus of the poles of a given straight line with respect to a confocal system of conics.**

Let $$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \qquad \text{(A)}$$
be one of the confocal system of conics, and
$$lx+my=1$$
the given straight line.

Suppose $(x_1 y_1)$ is the pole with respect to the conic (A). Then the given line must be the same as the polar of this point, *i.e.*, as
$$\frac{xx_1}{a^2+\lambda}+\frac{yy_1}{b^2+\lambda}=1. \qquad (\S\ 189)$$

Hence, comparing coefficients, we have
$$\frac{x_1}{a^2+\lambda}=l, \quad \frac{y_1}{b^2+\lambda}=m.$$
$$\therefore \quad x_1=(a^2+\lambda)\,l, \quad y_1=(b^2+\lambda)\,m.$$

To find the locus of the poles we must eliminate $\lambda$ between these equations.

Now $$\frac{x_1}{l}-\frac{y_1}{m}=a^2+\lambda-(b^2+\lambda)=a^2-b^2.$$

Thus the equation of the locus is
$$\frac{x}{l}-\frac{y}{m}=a^2-b^2.$$

Consequently the locus required is a straight line perpendicular to the given straight line.

Now the given straight line touches one of the confocal conics. Suppose $P$ is the point of contact, and $PT$ the given line. Then $P$ is the pole with respect to the touching confocal. Hence the locus required is the straight line $PG$ through $P$ at right angles to $PT$.

Cor. If $PG$ is the locus of the poles of $PT$, then $PT$ is the locus of the poles of $PG$.

Since the two confocals through $P$ cut at right angles, and $PT$ is the tangent to one, therefore $PG$ is the tangent to the other. Hence $P$ is one of the poles of $PG$, and, as the locus is perpendicular to the line $PG$, it must be the line $PT$.

**Exercises.**

32. Find the locus of the poles of the line $x+y=5$ with respect to the conics confocal with $2x^2+3y^2=6$.

33. If $l_1x+m_1y=1$ be the locus of the poles of $lx+my=1$ with respect to the confocal system $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$, show that
$$(a^2-b^2)\,ll'=(b^2-a^2)\,mm'=1.$$

34. Show from Ex. 33 that the relation between the two lines is reciprocal.

### 251. Confocal parabolas.

We have hitherto confined our attention to confocal central conics. We shall, in conclusion, say a few words as to confocal parabolas.

Since a parabola has only one focus, matters are rather different here. We say that

Two **parabolas** are **confocal** when they have the same focus and the same axis.

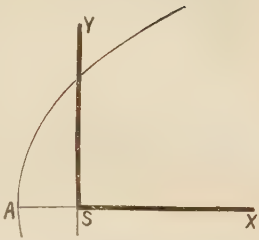


Fig. 90.

### 252. Equation of a confocal system of parabolas.

Take the common focus $S$ for origin, and the axis for axis of $x$. Then we know that, referred to its vertex $A$ as origin, the equation of a parabola is of the form

$$y^2 = 4\lambda x \quad \text{where} \quad \lambda = SA.$$

Thus, changing the origin to $S$, the equation becomes

$$y^2 = 4\lambda(x+\lambda) \quad \ldots\ldots\ldots\ldots \quad (5),$$

and this is the general equation required.

We leave the following properties as examples to the reader; the process in the analogous property for the central conics will suggest methods of attack.

#### Exercises.

35. Two confocal parabolas can be drawn through any given point.

36. The two confocal parabolas through any point have their concavities in opposite directions.

[This follows from the fact that the roots of the quadratic in $\lambda$ are of unlike sign.]

37. Two confocal parabolas cut at right angles.

38. The locus of the pole of the straight line $lx + my = 1$, with respect to the confocal parabolas $y^2 = 4\lambda(x+\lambda)$, is the straight line

$$mx - ly + m/l = 0.$$

[Let $(x_1, y_1)$ be the pole of the line with respect to $y^2 = 4\lambda(x+\lambda)$; then the given line must be the same as $yy_1 - 2\lambda x = 2\lambda(x_1 + 2\lambda)$. Hence see that

$$x_1 + 2\lambda = -\frac{1}{l} \quad \text{and} \quad 2\lambda = -\frac{ly_1}{m}.]$$

39. If, of two lines, the first is the locus of the pole of the second, then the second is the locus of the pole of the first.

## MISCELLANEOUS EXERCISES ON CHAP. XVIII.

40. If $A, B, C, D$ are the four points of intersection of two rectangular hyperbolas, show that the pairs of lines $BC, AD$; $CA, BD$; $AB, CD$ are at right angles, and hence that the four points form a triangle and its orthocentre.

41. $ABC$ is a triangle, $AD$ the perpendicular from $A$ on $BC$, and $P$ the orthocentre. Show that $DP . DA = DB . DC$. Hence, taking $D$ for origin, deduce that **all rectangular hyperbolas through the angular points of a triangle pass also through the orthocentre of the triangle.**

42. The locus of the centres of the rectangular hyperbolas in Ex. 41 is a circle passing through the middle points of the lines $BC$, $CA$, $AB$, $AP$, $BP$, $CP$, and through the feet of the perpendiculars.

[This circle is known as the *nine-point circle.*]

43. The locus of the centres of all conics through four points on a circle is a rectangular hyperbola.

44. Show that the equation of the polar of the origin with respect to the conic $\quad (x/\alpha + y/\gamma - 1)(x/\beta + y/\delta - 1) + \lambda xy = 0$

is $\quad x(1/\alpha + 1/\beta) + y(1/\gamma + 1/\delta) - 2 = 0.$

45. Find the general equation of a conic circumscribing the triangle formed by $x = 2$, $y + 1 = 0$, $x + y = 0$.

46. Find the equation of the circle circumscribing the triangle of Ex. 45.

47. A given ellipse moves so as always to touch two fixed rectangular lines. Find the locus of its centre.

48. Find the coordinates of the centre of the conic

$$(x/m + y/n - 1)^2 = 2kxy.$$

49. If $S = 0$ and $S' = 0$ be the equations of two conics, then the polar of a point $(x_1, y_1)$ with respect to $S + \lambda S' = 0$ is $u + \lambda u' = 0$, where $u = 0$ is the polar with respect to $S$, and $u' = 0$ the polar with respect to $S'$.

50. Exhibit, by a diagram, the system of conics $ax^2 + 2\lambda xy + by^2 = 1$, where $a$ and $b$ are constant and $\lambda$ is variable.

51. Find the coordinates of the points in which the ellipse $x^2/a^2 + y^2/b^2 = 1$ intersects the confocal rectangular hyperbola.

52. Show that, if tangents are drawn to a system of confocal conics from a fixed point on the major axis, the locus of the points of contact is a circle whose centre is on the major axis.

53. The difference between the squares of the central perpendiculars on two parallel tangents to two confocal conics is constant.

54. The polars of a point $P$ with respect to a system of conics through four fixed points all pass through another point $Q$. Further, the polars of $Q$ all pass through $P$.

55. If $$ax^2+2hxy+by^2+2gx+2fy+c=0$$
and $$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$$
be the equations of the pairs of opposite sides of a quadrilateral inscribed in a circle, show that $H(b-a)=h(B-A)$.

56. If two conics have two points of contact, they cannot meet in any other point.

57. Find the locus of the points of contact of tangents drawn in a fixed direction to a system of confocal conics.

58. If $S=0$, $S'=0$ denote the circles $x^2+y^2+2gx+2fy+c=0$, $x^2+y^2+2g'x+2f'y+c'=0$, the values of $k$, for which $S+kS'=0$ denotes a point, are roots of the quadratic
$$(g^2+f^2-c)+k(2gg'+2ff'-c-c')+k^2(g'^2+f'^2-c')=0.$$

59. Find the condition that the straight line $x/\alpha+y/\beta=1$ should touch the parabola which itself touches the axes at the points $(a,0)$, $(0,b)$.

60. From Ex. 44, show that, if $A$, $B$, $C$, $D$ be four points on a conic, and $AB$, $CD$ meet in $E$, $CA$, $BD$ in $F$, $CB$, $AD$ in $G$, then the polar of $E$ with respect to the conic passes through both $F$ and $G$. Deduce that the triangle $EFG$ is such that each side is the polar of the opposite vertex (*a self-conjugate triangle*).

61. Prove that the locus of the centres of conics passing through the four points of intersection of two equilateral hyperbolas is a circle.

62. If any two parallel tangents to an ellipse meet a fixed circle concentric with the ellipse, prove that the other two sides of the rectangle touch a confocal conic.

63. Prove that the locus of the point whose coordinates are given by $x=a\cos^4\theta$, $y=b\sin^4\theta$ is part of a parabola touching the coordinate axes; and find the equation of the tangent at a point for which $\theta$ is given.

64. If the two confocal ellipses
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{and} \quad \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$
be cut by the straight line $x\cos\theta+y\sin\theta=p$, and if $T$ and $T'$ be the poles of this line with respect to the two ellipses, prove that
$$TT'=\lambda/p.$$

65. Prove that the polars of a point $(x', y')$ with respect to the confocals $$x^2/(a^2+\lambda)+y^2/(b^2+\lambda)=1$$
touch the conic $$\sqrt{xx'}+\sqrt{-yy'}+\sqrt{a^2-b^2}=0.$$

# CHAPTER XIX.

## ENVELOPES.

253. **Envelopes.** — Suppose the coefficients in the equation of a curve depend on a quantity $\mu$ which is capable of variation; then when we give $\mu$ a definite value we get a perfectly definite curve, and by varying $\mu$ we get a system of curves.

Perhaps some examples may assist the reader in grasping this idea.

The equation of the straight line

$$(ax+by+c)+\mu\,(a'x+b'y+c)=0$$

depends on the quantity $\mu$. When $\mu$ varies we get a whole system or family of straight lines which, as we know, all pass through a fixed point (Part I., § 28).

Again, from the equation

$$y=\mu x+\frac{a}{\mu}$$

we get a series of straight lines by allowing $\mu$ to vary, and each line of the series touches the parabola $y^2=4ax$ (§ 48).

As another example, the equation

$$\frac{x^2}{a^2+\mu}+\frac{y^2}{b^2+\mu}=1$$

represents a system of confocal conics.

**Parameter.** — Definition. The quantity $\mu$ is called a variable **parameter**, and the system of curves is said to depend on one parameter.

254. **Ultimate intersections.**—If we give $\mu$ a definite value $\mu_1$ we obtain a perfectly definite curve. If now we give to $\mu$ a value only differing very slightly from $\mu_1$, say $\mu_1+\epsilon$ where $\epsilon$ is very small, we get another curve, only very slightly displaced from the former. These two curves meet in a certain number of points, and when $\epsilon$ is made indefinitely small such points are called ultimate intersections of the curves.

Thus, in the system

$$y = \mu x + a/\mu$$

we have a definite line

$$y = \mu_1 x + a/\mu_1,$$

and the one obtained by giving to $\mu$ a value slightly different from $\mu_1$ is another line very close to the former, viz.,

$$y = (\mu_1+\epsilon)\,x + a/(\mu_1+\epsilon).$$

Now both these lines are tangents to the parabola $y^2 = 4ax$; hence, as two infinitely close tangents to a curve ultimately meet on the curve, we infer that in this case the ultimate intersections all lie on the parabola which is touched by the lines. We shall presently see that this is true in general.

255. **Locus of ultimate intersections.**

If we take every curve of our system and find its points of intersection with the curve of the system indefinitely near it, we get an infinite number of points of intersections all lying on a curve which is the locus of ultimate intersections.

Thus, in Fig. 91, suppose the curved lines 1, 2, 3, 4, 5 are portions of five very close curves in the system. Suppose 1 meets 2 in $P$, 2 meets 3 in $Q$, 3 meets 4 in $R$, 4 meets 5 in $S$; then $P$, $Q$, $R$, $S$ are all on the locus of ultimate intersections when we make the four curves come infinitely close together.

256. **Each original curve touches the locus of ultimate intersections.**

Consider the curve 2 (Fig. 91). It meets the locus of ultimate intersections, indicated by the dotted line,

(i.) where it meets 1, *i.e.* in $P$;

(ii.) where it meets 3, *i.e.* in $Q$.

Now when the three curves 1, 2, 3 approach indefinitely close to each other so also will $P$ and $Q$; thus the locus of ultimate intersections meets the curve 2 in two coincident points, and hence touches it.

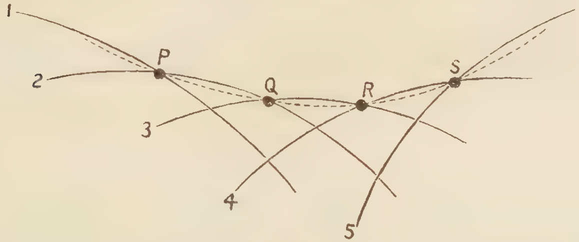



Fig. 91.

Similarly the locus of ultimate intersection touches every one of the original curves, and on that account is called the **envelope** of the system.

*Example.*—Let us consider *ab initio* the system of straight lines

$$y = \mu x + \frac{a}{\mu} \quad \text{or} \quad \mu^2 x - \mu y + a = 0.$$

Giving $\mu$ two definite values, $\mu_1$ and $\mu_2$, we obtain two straight lines, viz., $\mu_1^2 x - \mu_1 y + a = 0$ and $\mu_2^2 x - \mu_2 y + a = 0$.

Hence at their point of intersection we obtain on subtraction

$$(\mu_1^2 - \mu_2^2)\, x - (\mu_1 - \mu_2)\, y = 0 \quad \text{or} \quad (\mu_1 + \mu_2)\, x - y = 0.$$

Now, if we suppose $\mu_2$ to become indefinitely near to $\mu_1$ in value, the last becomes $2\mu_1 x - y = 0$, which is satisfied by the coordinates of the point of ultimate intersection.

(Note that dividing across by $\mu_1 - \mu_2$ enables us to put $\mu_2 = \mu_1$ without making the left-hand side vanish identically.)

But $\mu_1^2 x - \mu_1 y + a = 0$.

Hence to find the locus of ultimate intersection we must eliminate $\mu_1$ between $\mu_1^2 x - \mu_1 y + a = 0$ and $2\mu_1 x - y = 0$.

From the second equation we have

$$\mu_1 = \frac{y}{2x};$$

hence

$$\frac{y^2}{4x^2} x - \frac{y}{2x} y + a = 0;$$

*i.e.*,

$$y^2 - 4ax = 0.$$

This is the result we should have expected as indicated in the previous articles.

**257. To find the envelope of the curve whose equation is** $\mu^2 P + \mu Q + R = 0$,
**where $P$, $Q$, $R$ are definite functions of $x$ and $y$, and $\mu$ is a variable parameter.**

We shall find the locus of ultimate intersections.

Two curves of the system are

$$\mu_1^2 P + \mu_1 Q + R = 0 \quad \text{and} \quad \mu_2^2 P + \mu_2 Q + R = 0 \ldots \text{(A)}.$$

The coordinates of their points of intersection satisfy both equations (A), and hence they satisfy

$$P(\mu_1^2 - \mu_2^2) + Q(\mu_1 - \mu_2) = 0,$$

as we find on subtraction, or

$$P(\mu_1 + \mu_2) + Q = 0.$$

If now we make $\mu_2$ differ from $\mu_1$ by an indefinitely small quantity, their points of intersections become points on the envelope. Making therefore $\mu_2 = \mu_1$, ultimately we get

$$2\mu_1 P + Q = 0.$$

Thus to find the locus we have to eliminate $\mu_1$ between

$$2\mu_1 P + Q = 0 \quad \text{and} \quad \mu_1^2 P + \mu_1 Q + R = 0.$$

Also

$$\mu_1 = -\frac{Q}{2P},$$

$$\therefore \quad \frac{Q^2}{4P^2} P - \frac{Q}{2P} Q + R = 0$$

*i.e.*

$$\boldsymbol{Q^2 = 4PR} \quad \ldots\ldots\ldots\ldots \quad \textbf{(1)}.$$

Hence we have the following

RULE.—*The equation of the envelope is obtained by putting down the condition that the quadratic equation in the variable parameter should have equal roots.*

258. There is another very interesting way of interpreting the result of the last article.

Consider the curves of the system which pass through a given point. There are clearly two of them, for we must have

$$\mu^2 P + \mu Q + R = 0$$

when the coordinates $x_1, y_1$ are substituted in $P$, $Q$, and $R$, and, as this is a quadratic for $\mu$, there are two curves through the point $(x, y)$.

Now, if the two curves through $(x_1, y_1)$ coincide, the point $(x_1, y_1)$ lies on the envelope, and the condition that the curves coincide is that the two values of $\mu$ found from the quadratic in $\mu$ should be equal.

The condition for this is

$$Q^2 = 4PR,$$

which is the same condition as in § 257.

*Example.* —*Find the envelope of the polars of a given point with respect to a system of confocal conics.*

Let any one of the confocal system be

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

the polar of $(x_1, y_1)$ with respect to this is

$$\frac{xx_1}{a^2+\lambda} + \frac{yy_1}{b^2+\lambda} = 1.$$

Now, to find the envelope, we must regard $\lambda$ as a variable parameter. The equation is

$$(a^2+\lambda)(b^2+\lambda) - (b^2+\lambda)xx_1 - (a^2+\lambda)yy_1 = 0$$

or
$$\lambda^2 + \lambda(a^2+b^2-xx_1-yy_1) + a^2b^2 - b^2xx_1 - a^2yy_1 = 0;$$

and hence, by the rule, the envelope is

$$(a^2+b^2-xx_1-yy_1)^2 = 4(a^2b^2-b^2xx_1-a^2yy_1).$$

Since the terms of the second degree are $(xx_1+yy_1)^2$, this represents a parabola. Hence the envelope required is a parabola.

**Exercises.**

1. Two lines of the system $m^2x - my + a = 0$ can be drawn through any point, and they coincide if the point be on the parabola $y^2 = 4ax$, which is the envelope.

2. Prove that the envelope of the family of straight lines obtained by giving different values to $\mu$ in the equation

$$\mu(Ax+By) + \frac{1}{\mu}(Bx-Ay) + 1 = 0$$

is a rectangular hyperbola, and find the lengths and positions of its axes.

3. A line moves so that the product of its intercepts on the axes is constant. Show that its envelope is a hyperbola having the axes for asymptotes.

[Take one of the intercepts for variable parameter, and express the other in terms of it by means of the given condition.]

4. $P$ is a point on a parabola, $PM$ its ordinate. Find the envelope of the diagonal $MQ$ of the parallelogram $PMAQ$.

259. Sometimes it is necessary to find the envelope of a curve whose equation contains *two* variable parameters, these two parameters being connected by a further equation. In simple cases we can obtain the envelope by *eliminating one of the parameters from the equation of the curve by means of the equation connecting the parameters.* Other methods are sometimes applicable, as, *e.g.*, in Example (iii.) below.

*Examples.* — (i.) *Find the envelope of the circle* $(x-c)^2+y^2=d^2$, *where* $c^2+d^2=k^2$.

Eliminating $d$, we get

$$(x-c)^2+y^2=k^2-c^2 \quad \text{or} \quad 2c^2-2c\,.\,x+y^2-k^2=0.$$

The envelope is got by making the roots of this equation in $c$ equal. Thus the envelope is

$$x^2=2(y^2-k^2) \quad \text{or} \quad x^2-2y^2+2k^2=0,$$

which is evidently a hyperbola.

(ii.) *Find the envelope of a straight line the sum of whose intercepts on the coordinate axes is constant.*

Let $c$ and $d$ be the intercepts, $k$ their *constant* sum. Then

$$c+d=k,$$

and the equation of the line is $x/c+y/d=1$.

Eliminating $c$ from these, we get $x/(k-d)+y/d=1$.

or $\quad d(k-d)-dx-(k-d)y=0 \quad$ or $\quad d^2-(k-x+y)d+ky=0.$

The envelope is got by making the two values of the variable parameter $d$ equal, and is therefore

$$(k-x+y)^2=4ky \quad \text{or} \quad x^2-2xy+y^2-2kx-2ky+k^2=0,$$

which, since the terms of the second degree in $x$ and $y$ form a perfect square, represents a parabola.

[Compare this proof with that given in § 261.]

(iii.) **The envelope of the curve**

$$P\cos\theta+Q\sin\theta=R,$$

**where $P$, $Q$, $R$ are functions of the coordinates and $\theta$ is the variable parameter, is the curve given by**

$$P^2+Q^2=R^2.$$

Let $t=\tan\frac{1}{2}\theta$. Then

$$\cos\theta=\frac{1-t^2}{1+t^2},\quad \sin\theta=\frac{2t}{1+t^2};$$

$$\therefore \quad P\frac{1-t^2}{1+t^2} + Q\frac{2t}{1+t^2} = R$$

or $$P(1-t^2) + 2t \, . \, Q = R(1+t^2),$$

*i.e.* $$t^2(P+R) - 2t \, . \, Q + (R-P) = 0.$$

The equation of the envelope is the condition that the parameter has equal roots.

It therefore is $$Q^2 = (P+R)(R-P)$$

or $$P^2 + Q^2 = R^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (\mathbf{2}).$$

This result is worth remembering, although, as a rule, the student should in each case work through the process exemplified above.

### Exercises.

5. Apply the method of envelopes to show that the straight line $x\cos\theta/a + y\sin\theta/b = 1$ touches the curve $x^2/a^2 + y^2/b^2 = 1$, $\theta$ being the variable parameter.

6. Prove that the envelope of the line $ax\cos\theta + by\sin\theta = c$ is $a^2x^2 + b^2y^2 = c^2$.

7. A straight line moves so that the foot of the perpendicular on it from a fixed point lies on a fixed straight line. Find its envelope.

[Take the given point for origin, and let the fixed line be $x+a=0$. Find the equation of the line joining the point of intersection of $lx+my+1=0$ and $x+a=0$ with the origin, and then put down the condition that this line should be perpendicular to $lx+my+1=0$. The result is a parabola. See § 145.]

8. $OX$ and $OY$ are two fixed lines, and a variable line $AB$ cutting them in $A$ and $B$ moves so that $\lambda . OA + \mu . OB$ is constant. Show that its envelope is a parabola.

260. **Envelopes in general.**—It should be noted that the discussion of envelopes given in §§ 253–259 applies to curves generally and is not limited to those whose equations are of the second degree.

Further, it is clear that, if in the equation of the curve the parameter $\mu$ appears in powers *higher than the second*, an envelope can be found (theoretically, at least) from the condition that two of the roots in $\mu$ are equal.

*Example.—Find the envelope of the normals to the parabola* $y^2 = 4ax$.

The normal may be written

$$y = mx - 2am - am^3 \quad \text{or} \quad am^3 + m(2a-x) + y = 0.$$

By the theory of equations, if $m_1, m_2, m_3$ are the roots of this equation, we have

$$m_1 + m_2 + m_3 = 0,$$
$$m_1m_2 + m_1m_3 + m_2m_3 = (2a-x)/a,$$
$$m_1m_2m_3 = -y/a.$$

For the envelope, two of the roots in $m$ are equal, say, $m_1 = m_2$.

Then
$$2m_1 + m_3 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)},$$
$$m_1^2 + 2m_1 m_3 = (2a-x)/a \quad\ldots\ldots\ldots\ldots \text{(2)},$$
$$m_1^2 m_3 = -y/a \quad\ldots\ldots\ldots\ldots\ldots \text{(3)}.$$

Eliminating $m_3$ by means of (1), we get
$$m_1^2 - 4m_1^2 = (2a-x)/a \quad\text{or}\quad m_1^2 = (x-2a)/3a,$$
$$-2m_1^3 = -y/a \quad\text{or}\quad m_1^3 = y/2a.$$

Eliminating $m_1$, we get, as equation of envelope,
$$\left(\frac{x-2a}{3a}\right)^3 = m_1^6 = \left(\frac{y}{2a}\right)^2 \quad\text{or}\quad 4\,(x-2a)^3 = 27y^3.$$

NOTE.—The envelope of the system of normals to any curve is called the **evolute** of that curve.

### Exercise.

9. Find the envelope of the straight line $\mu^3 x - \mu^2 y + c = 0$, $\mu$ being the variable parameter.

261. **If in the straight line $lx + my + 1 = 0$ the coefficients $l$, $m$ be connected by a relation**
$$al^2 + 2hlm + bm^2 + 2gl + 2fm + c = 0,$$
**then the envelope of the line is a conic section.**

Let us first inquire how many lines of the system pass through a given point $(x_1, y_1)$. We must have
$$lx_1 + my_1 + 1 = 0 \quad\text{and}\quad al^2 + 2hlm + bm^2 + 2gl + 2fm + c = 0.$$

Now, if we make the second equation homogeneous by means of the first, we obtain
$$al^2 + 2hlm + bm^2 - 2\,(gl + fm)(lx_1 + my_1) + c\,(lx_1 + my_1)^2 = 0. \text{ (Pt.I., § 38)}$$

This being a quadratic for the ratio $l : m$, it has two roots. Now when the ratio $l : m$ is known, the direction of the line is known; thus the lines through $(x_1, y_1)$ are in one of two directions, and hence there are in general two such lines.

To find the envelope, we have to find the condition that these two coincide, *i.e.* the above quadratic for the ratio $l : m$ must have equal roots.

Now the quadratic may be written
$$l^2\,(a - 2gx_1 + cx_1^2) + 2lm\,(h - gy_1 - fx_1 + cx_1 y_1) + m^2\,(b - 2fy_1 + cy_1^2) = 0;$$
hence the condition for equal roots is
$$(a - 2gx_1 + cx_1^2)\,(b - 2fy_1 + cy_1^2) = (h - gy_1 - fx_1 + cx_1 y_1)^2$$
or
$$(ab - h^2) + 2x_1\,(hf - bg) + 2y_1\,(gh - af) + x_1^2\,(bc - f^2) + y_1^2\,(ca - g^2) + 2x_1 y_1\,(fg - ch) = 0,$$
the terms of degree higher than the second in $x_1$ and $y_1$ cancelling out.

Thus the envelope is a conic whose equation is
$$(bc - f^2)\,x^2 + 2xy\,(fg - ch) + y^2\,(ca - g^2) + 2x\,(hf - bg) + 2y\,(gh - af) + (ab - h^2) = 0.$$

*Example* (i.) *A line moves so that the sum of its intercepts on two fixed lines is constant. To show that it always touches a parabola.*

Take the given lines to be axes of coordinates, and $lx+my-1=0$ as the equation of the line.

Then the intercepts on the axes are $1/l$ and $1/m$. Thus

$$1/l+1/m = \text{const.} = \alpha \text{ (say)}, \quad i.e. \quad \alpha lm - l - m = 0.$$

Thus the relation between $l$ and $m$ is of the second degree, and hence, by the foregoing, the line always touches a conic.

The directions of the two lines through the point $(x, y)$ are given by

$$\alpha lm-(l+m)(lx+my)=0 \quad \text{or} \quad l^2.x+lm(x+y-\alpha)+m^2.y=0.$$

The condition for coincidence is

$$(x+y-\alpha)^2=4xy \quad \text{or} \quad x^2-2xy+y^2-2\alpha x-2\alpha y+\alpha^2=0.$$

This clearly denotes a parabola. The equation may be reduced to the form $\qquad \sqrt{x}+\sqrt{y}+\sqrt{\alpha}=0.$

The parabola therefore touches the axes of coordinates.

*Example* (ii.) *A line moves so that the product of the perpendiculars from two fixed points on it is constant. To show that it envelops a conic.*

Take the line joining the given points as the axis of $x$, and let the given points $A$, $B$ be $(c, 0)$ $(-c, 0)$ the axes being rectangular.

The perpendiculars on the line $lx+my+1=0$ are

$$\frac{lc+1}{\sqrt{l^2+m^2}} \quad \text{and} \quad \frac{-lc+1}{\sqrt{l^2+m^2}}.$$

Thus we have $$\frac{1-l^2c^2}{l^2+m^2}=\text{const.}=b^2, \text{ say};$$

$$\therefore \quad b^2(l^2+m^2)+l^2c^2=1.$$

Since this is of the second degree in $l$ and $m$, the envelope is a conic.

To find the equation we must make the quadratic

$$b^2(l^2+m^2)+c^2l^2-(lx+my)^2=0$$

have equal roots in $l/m$. The quadratic is

$$l^2\left\{b^2+c^2-x^2\right\}-2lmxy+m^2(b^2-y^2)=0$$

and the equation of the envelope is

$$(b^2+c^2-x^2)(b^2-y^2)=x^2y^2 \quad \text{or} \quad x^2b^2+y^2(b^2+c^2)=b^2(b^2+c^2),$$

*i.e.* $$\frac{x^2}{b^2+c^2}+\frac{y^2}{b^2}=1.$$

Thus the envelope is an ellipse having $\sqrt{b^2+c^2}$ and $b$ for semi-axes and the given points for foci.

**Exercise.**

10. If the product of the perpendiculars be $(-b^2)$, show in like manner that the envelope is a hyperbola having the given points for foci.

## MISCELLANEOUS EXERCISES ON CHAP. XIX.

11. A straight line forms, with two given lines, a triangle of constant area. Show that its envelope is a hyperbola having the given lines for asymptotes.

12. A straight line moves so that the difference of its intercepts on the axis is constant. Show that its envelope is a parabola.

13. Find the envelope of a circle that rolls along the axis of $x$.

14. $CP$ and $CD$ are conjugate diameters of an ellipse. Find the envelope of the line joining the middle points of the ordinates of $P$ and $D$.

15. Find the envelope of the straight line

$$y = mx + a\sqrt{1+m^2} + c - mb,$$

where $m$ is variable.

16. A parabola touches the coordinate axes, the equation of the chord of contact being $ax + by = 1$. Prove, by the method of envelopes, that the line $ax\lambda + by = \lambda/(\lambda+1)$ always touches the curve whatever be $\lambda$.

17. Show that the lines given by the equation

$$(x\cos\alpha + y\sin\alpha)^2 = A\cos^2\alpha + B\sin^2\alpha,$$

for different values of $\alpha$, all touch the same conic.

18. Deduce, from Ex. 17, the locus of the intersection of two tangents to a fixed conic, at right angles to each other.

19. If two tangents to an ellipse intersect on the circumference of a concentric circle, prove that their chord of contact touches another ellipse.

20. A line moves so that the sum of the squares of the perpendiculars on it from two fixed points is constant. Show that its envelope is an ellipse.

21. If the sum of the squares of the perpendiculars on to a straight line from any number of points be constant, the envelope of the line is an ellipse.

22. The tangents at points $P$ and $Q$ on an ellipse are at right angles. Show that $PQ$ touches a fixed concentric ellipse.

23. The envelope of a chord of an ellipse whose middle point lies on a fixed line is a parabola.

24. If a page of a book is turned down so that one corner moves along an opposite side of the page, show that the envelope of the line of crease is a parabola.

# CHAPTER XX.

## HARMONIC SECTION.

262. **Harmonic range.**—DEFINITION.—Any number of points in a straight line are said to constitute a **range** of points.

Four points $A$, $B$, $C$, $D$ are said to form a **harmonic range** when two of them divide the distance between the other two internally and externally in the same ratio.

Then $$\frac{AC}{CB}=\frac{AD}{BD} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Suppose, for example, that $C$ and $D$ divide the segment $AB$ internally and externally in the same ratio; then

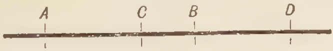



Fig. 92.

$A$, $B$, $C$, $D$, form a harmonic range and the pair of points $C$, $D$ are said to be **harmonically conjugate** with respect to $A$ and $B$.

**263. If $C$ and $D$ are harmonically conjugate with respect to $A$ and $B$, then $A$ and $B$ are harmonically conjugate with respect to $C$ and $D$.**

For we have, by hypothesis,

$$\frac{AC}{CB}=+\frac{AD}{BD},$$

with the usual convention as to sign; hence

$$\frac{AC}{AD}=+\frac{CB}{BD},$$

so that $A$ and $B$ divide $CD$ internally and externally in the same ratio, which proves the proposition.

COR. The relation between the points $A$, $B$, $C$, $D$ can be written

$$\frac{AC \cdot BD}{AD \cdot BC} = -1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(2)},$$

for

$$\frac{AC \cdot BD}{AD \cdot CB} = 1 \quad \text{and} \quad BC = -CB.$$

This result should be remembered. To get the numerator, we write down the four points *in the order in which they occur on the line;* and, for the denominator, we take the same point first, and the others in exactly the reverse order, thus

$$\left.\begin{matrix} A, C, B, D \\ A, D, B, C \end{matrix}\right\}.$$

In consequence of the above reciprocal relation, we shall frequently say that two such pairs of points are **harmonic.**

264. **To find the relation connecting the distances of two harmonic pairs of points from any point of the line.**

Suppose the point from which all the distances are

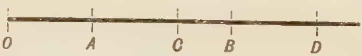


Fig. 93

measured is $O$, and that the lengths of $OA$, $OB$, $OC$, $OD$ are $x_1$, $x_2$, $x_3$, $x_4$ respectively. (Of course, distances in one direction are measured positively, and in the other negatively.)

Then

$$\frac{AC}{CB} = \frac{AD}{BD}.$$

Now $\quad AC = x_3 - x_1, \quad CB = x_2 - x_3,$ &c.,

so that we get

$$\frac{x_3 - x_1}{x_2 - x_3} = +\frac{x_4 - x_1}{x_4 - x_2},$$

for, in general, a distance such as $AD$ is found by subtracting the $x$ belonging to $D$ from that belonging to $A$, and so on.

Hence, from the above equation, we have

$$(x_3 - x_1)(x_4 - x_2) - (x_4 - x_1)(x_2 - x_3) = 0,$$

or, on multiplying out and collecting terms,

$$2(x_1x_2+x_3x_4)-x_1x_4-x_2x_3-x_1x_3-x_2x_4=0;$$

$$\therefore \quad \mathbf{2}\,(x_1x_2+x_3x_4)=(x_1+x_2)(x_3+x_4) \quad \ldots\ldots \ (\mathbf{3}).$$

COR. Conversely, when this relation holds, we have

$$\frac{x_3-x_1}{x_2-x_3}=-\frac{x_4-x_1}{x_2-x_4},$$

and the two pairs of points $A$, $B$ and $C$, $D$ are harmonic.

Thus, if the relation (3) holds for one position of the point $O$, the four points are harmonic, and it holds for all.

265. **Particular cases.**—By supposing the point $O$ to have particular positions, we can deduce two very important relations from the last article.

I. Let $O$ coincide with $A$; then $x_1=0$, and the relation becomes

$$2x_3x_4=x_2(x_3+x_4) \quad \text{or} \quad \frac{2}{x_2}=\frac{1}{x_3}+\frac{1}{x_4},$$

so that $AB$ is the harmonic mean between $AC$ and $AD$. This shows the connection between harmonic means and harmonic ranges.

Thus $AC$, $AB$, $AD$ are in harmonic progression.

II. Suppose $O$ is the middle point of $AB$; then $x_1=-x_2$, so that $x_1+x_2=0$, and the right-hand side of (3) is zero.

Hence $\qquad x_1x_2+x_3x_4=0 \quad \text{or} \quad x_3x_4=x_1^2.$

Consequently, if $C$ and $D$ be harmonically conjugate with respect to $A$ and $B$, and $O$ be the middle point of $AB$, we have

$$OC \,.\, OD = OA^2 = OB^2.$$

Clearly $C$ and $D$ are on the same side of $O$; otherwise $OC \,.\, OD$ would be negative.

This gives an easy method of finding a point harmonically conjugate to $C$ with respect to $A$ and $B$.

*Example.—The distances of four points on a line from a fixed point on it are 5, $7\frac{1}{4}$, 8, $9\frac{1}{2}$ units respectively. Find whether or not any pair of them is harmonically conjugate to the other pair.*

The easiest way to do this is to find whether the distances from any one can be so arranged as to be in harmonic progression (*cf.* I.).

The distances from the first to the second, third, and fourth are $2\frac{1}{4}$, 3, $4\frac{1}{2}$. If these can be arranged in harmonic progression, their reciprocals $\frac{4}{9}$, $\frac{1}{3}$, $\frac{2}{9}$ can be arranged in arithmetic progression. But they are clearly in arithmetic progression as written down. Consequently the first and third of the points are harmonically conjugate with respect to the second and fourth, and *vice versa* (by I.).

266. **Particular case of harmonic range.—Point at infinity.**

If the two pairs of points $A, B$; $C, D$ are harmonic, then, when $C$ is the middle point of $AB$, $D$ is at infinity.

For, since $AC = CB$, we must have $AD = DB$, which can only happen when the quantities are both infinite, since their difference is finite.

Hence the point harmonically conjugate to the middle point of $AB$ is the point at infinity on the line (see §§ 199, 200).

This can also be easily seen from the fact that, if $O$ be the middle point of $AB$, then $OC \cdot OD = OA^2$ (§ 265), for the nearer $C$ comes to $O$ the further $D$ recedes from it, and when $C$ finally coincides with $O$ $D$ is at infinity.

We may state this important result in words, as follows :—

"Any two points on a line, the point midway between them, and the point at infinity on the line form a harmonic range."

**Exercises.**

1. Prove that $DC$ in § 265, Case I. is the harmonic mean between $DB$ and $DA$.

2. The distances of four points on a line from a fixed point on it are 3, 4, 5, $3\frac{3}{5}$ inches respectively. Examine whether any two of them are harmonically conjugate with respect to the other two.

3. The distances of four points $A, B, C, D$ from an origin $O$ on the line are $x_1, x_2, x_3, x_4$. Show that, if the relation

$$2(x_1 x_2 + x_3 x_4) = (x_1 + x_2)(x_3 + x_4)$$

is satisfied for one origin $O$, it is satisfied for all positions of $O$.

4. If $C$ and $D$ are harmonically conjugate with respect to $A$ and $B$, and $OA = 2$, $OB = 4$, mark the positions of $D$ for the positions of $C$ given by

$$OC = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ -1,\ -2,\ -3,\ -4,\ -5,\ -6.$$

5. If $O$ be the middle point of $AB$, and $OC \cdot OD = \frac{1}{4}AB^2$, show that $C$ and $D$ divide $AB$ internally and externally in the same ratio.

6. If $C$ and $D$ are harmonically conjugate to $A$ and $B$ (Fig. 93), show that $C$ and $D$ always move in opposite directions.

Of $C$ and $D$, that moves more rapidly which is the further away from $O$.

7. If $C$ move to $C'$ and $D$ to $D'$, where $CC'$ and $DD'$ are very small, show that $\dfrac{DD'}{CC'} = \dfrac{OD}{OC}$.

**267. The distances of two pairs of points from the origin $O$ being given by the quadratics**

$$a_1x^2+2b_1x+c_2=0 \quad \text{and} \quad a_2x^2+2b_2x+c_2=0,$$

**to find the condition that the two pairs may be harmonic.**

Let $x_1, x_2$ be the roots of the first quadratic and $x_3, x_4$ the roots of the second; then we know that

$$2(x_1x_2+x_3x_4)=(x_1+x_2)(x_3+x_4); \qquad (\S\ 264)$$

but, by the theory of quadratics,

$$x_1+x_2=-2b_1/a_1, \quad x_3+x_4=-2b_2/a_2,$$

$$x_1x_2=c_1/a_1, \quad x_3x_4=c_2/a_2, \text{ (\textit{Tut. Alg.} II., 156)}$$

and hence $\quad 2\left(\dfrac{c_1}{a_1}+\dfrac{c_2}{a_2}\right)=\left(-\dfrac{2b_1}{a_1}\right)\left(-\dfrac{2b_2}{a_2}\right)$

or

$$a_1c_2+a_2c_1=2b_1b_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (4).$$

COR. One pair of points can be found harmonic to each of two given pairs.

For let $a_1x^2+2b_1x+c_1=0$ and $a_2x^2+2b_2x+c_2=0$ be the given ones.

Then, if $Ax^2+2Bx+C=0$ be two points harmonic to the former pairs, $Ac_1+Ca_1-2Bb_1=0$, $Ac_2+Ca_2-2Bb_2=0$, from which the ratios of $A$, $B$, $C$ may be determined in the usual way. (*Tut. Alg.* II., § 68.)

### Exercises.

8. The distances of $A$ and $B$ from a fixed point $O$ on the line are given by
$$x^2-5x+3=0.$$
If $OC=1$, find $OD$ where $C$ and $D$ are conjugate to $A$ and $B$.

[Let $OD=a$; then the two quadratics $(x-1)(x-a)=0$, $x^2-5x+3=0$ must be harmonic.]

9. Find the value of $q$ in order that the pairs of points
$$x^2+2x-1=0 \quad \text{and} \quad x^2+4x+q=0$$
may be harmonic.

10. Find the pair of points which are harmonic to the two pairs given by $\quad x=1,\ x=3; \quad x=4,\ x=6.$

11. The pair of points harmonic to both $a_1x^2+2b_1x+c_1=0$ and $a_2x^2+2b_2x+c_2=0$ is given by
$$(a_1x+b_1)(b_2x+c_2)-(a_2x+b_2)(b_1x+c_1)=0.$$

**269. If two pairs of rays $O\alpha$, $O\beta$ and $O\gamma$, $O\delta$ be such that one straight line meets them in two pairs of points $A, B$ and $C, D$ which are harmonic, then every straight line meeting the rays is cut harmonically by them.**

$A$, $B$ and $C$, $D$ are harmonic points if

$$\frac{AC.BD}{AD.BC} = -1. \qquad (\S\ 263)$$

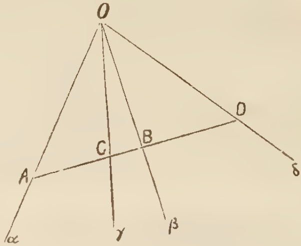


Fig. 94.

Let $p$ be the perpendicular from $O$ on the given line; then we have

$$AC.p = 2\triangle AOC$$
$$= OA.OC \sin AOC,$$

and therefore

$$AC = OA.OC \sin AOC/p,$$

with similar values for $BD$, $AD$, $BC$.

Hence, substituting in the equation above, we have

$$\frac{\dfrac{OA.OC \sin AOC}{p} \cdot \dfrac{OB.OD \sin BOD}{p}}{\dfrac{OA.OD \sin AOD}{p} \cdot \dfrac{OB.OC \sin BOC}{p}} = -1,$$

whence $$\frac{\sin AOC . \sin BOD}{\sin AOD . \sin BOC} = -1 \ldots\ldots\ldots\ldots\ldots \quad \textbf{(5)}.$$

Now this condition does not involve the position of the cutting line at all, but depends only on the mutual inclinations of the rays, and therefore, if it hold for one cutting line, it holds for all.

270. **Pencil.**—Definition.—A number of straight lines passing through the same point are said to form a **pencil.**

The point is called the *vertex*, and each individual line is often called a *ray* of the pencil.

Thus $OA$, $OC$, $OB$, $OD$ in Fig. 94, form a pencil. Such a pencil is sometimes indicated by the notation $O(ACBD)$.

**Harmonic pencil.**—Definition.—If two pairs of lines

$O\alpha$, $O\beta$ and $O\gamma$, $O\delta$ meet any (and therefore by the above proposition *all*) cutting lines in harmonic points, the pairs of lines are said to be harmonic, and the four lines $O\alpha$, $O\beta$, $O\gamma$, $O\delta$ form a **harmonic pencil.**

**Projection.**—DEFINITION.—If we join any point $P$ of one line to a fixed point $O$ and produce $OP$ to meet another line in $P'$, then $P'$ is called the projection of $P$.

COR. I. It clearly follows that, if four points form a harmonic range, then their projections on any other line do also.

COR. II. **The pencil formed by joining *any* vertex to the points of a harmonic range is harmonic.**

271. **Particular case.**

From the result of the last article we can deduce one which is often of great use in examining whether a given pencil be harmonic or not.

Suppose the cutting line $ABCD$ is parallel to $O\delta$; then $D$ is at infinity, and hence $C$ must be the middle point of $AB$. Thus,

*If four lines form a harmonic pencil, the portion of a line parallel to one of them intercepted between two of the others is bisected by the third, and conversely.*

For example, the portion of a tangent intercepted between the asymptotes of a hyperbola is bisected at the point of contact, and the tangent is parallel to the conjugate diameter; consequently any two conjugate diameters of a hyperbola are harmonically conjugate with respect to the asymptotes.

This can also be seen by analysis, for in the hyperbola

$$Ax^2+2Hxy+By^2=1$$

the asymptotes are

$$Ax^2+2Hxy+By^2=0,$$

and the lines $ax^2+2hxy+by^2=0$ are conjugate diameters if

$$aB+bA-2hH=0 \quad \text{(Ex. 10, p. 175).}$$

But this (see § 272) is the condition that the lines should be harmonically conjugate with respect to the asymptotes.

272. **To find the condition that the two pairs of lines given by the equations**

$$a_1x^2+2h_1xy+b_1y^2=0 \quad \textbf{and} \quad a_2x^2+2h_2xy+b_2y^2=0$$

**should be harmonic.**

Suppose the first pair of lines are $O\alpha$, $O\beta$ and the second pair $O\gamma$, $O\delta$, and further that the line $y=1$ meets $OY$ in $O'$ and the pairs of lines in $A$, $B$, $C$, $D$ respectively. Then $A$, $B$ and $C$, $D$ must be harmonic.

To find $O'A$, $O'B$ we have to make $y=1$ in the first equation; thus they are the roots of

$$a_1x^2+2h_1x+b_1=0.$$

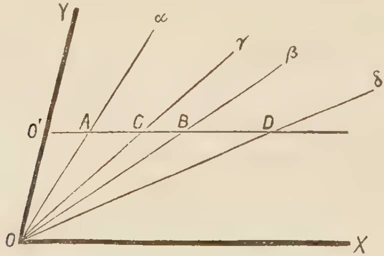



Fig. 95.

Similarly, the second pair $O'C$, $O'D$ are the roots of

$$a_2x^2+2h_2x+b_2=0,$$

and, consequently, by a known result, § 267, the condition is

$$\boldsymbol{a_1b_2+a_2b_1-2h_1h_2=0} \quad\text{..............} \quad \textbf{(5).}$$

Cor. I. Suppose that the second pair of lines are the axes, so that

$$a_2=b_2=0,$$

and the condition reduces to $h_1h_2=0$. But $h_2$ is not zero; consequently $h_1=0$.

Hence the pair of lines $ax^2+by^2=0$ are always harmonic to the axes.

It follows at once that the lines $y-mx=0$ and $y+mx=0$ are harmonic, for their equation is

$$y^2-m^2x^2=0,$$

in which there is no term in $xy$.

Cor. II. **If two pairs of lines be harmonic and one pair be at right angles, they bisect the angles between the other two.**

Take the perpendicular pair of lines as axes of coordinates; then the others are given by

$$ax^2+by^2=0,$$

so that they are clearly equally inclined to the axes.

**Exercises.**

12. If $ABC$ be a triangle and $D$ the middle point of $BC$, then $AB$, $AD$, $AC$ and the line through $A$ parallel to the base form a harmonic pencil.

13. **Two lines and the lines bisecting the angles between them form a harmonic pencil.**

[This follows for any cutting line by Euclid VI. 3. Or a line parallel to one bisector cuts off an isosceles triangle, and, as the bisectors are at right angles, the other bisects the base.]

14. **Given three lines $Oa$, $Ob$, $Oc$, prove the truth of the following construction for $Od$ such that $Oa$, $Ob$ and $Oc$, $Od$ are harmonically conjugate. Take any point $P$ on $Oc$ and draw $PM$, $PN$ parallel to $Oa$, $Ob$ meeting $Ob$, $Oa$ in $M$ and $N$ respectively, and then draw $Od$ parallel to $MN$.**

15. A variable straight line meets *any* four fixed lines $O\alpha$, $O\beta$, $O\gamma$, $O\delta$ in points $A$, $B$, $C$, $D$. Show that the value of $\dfrac{AB\,.\,CD}{AD\,.\,CB}$ is the same for all positions of the cutting line.

16. Prove that the pair of lines $y=3x$, $y=4x$ and the pair $y=5x$, $3y=11x$ form a harmonic pencil.

[Show that the points in which they meet $x=1$ form a harmonic range.]

17. Show that the pairs of lines

$$x^2+2xy-y^2=0 \quad \text{and} \quad 3x^2-4xy-y^2=0$$

form a harmonic pencil.

18. Whether the axes are rectangular or oblique the straight lines $y=mx$, $y=-mx$ form a harmonic pencil with the axes.

19. Find the value of $\lambda$ in order that the pairs of lines

$$3x^2+xy-y^2=0, \quad x^2+\lambda xy+y^2=0$$

should form a harmonic pencil.

20. The axes being rectangular, show that a pair of lines which form a harmonic pencil with the imaginary pair $x^2+y^2=0$ are at right angles.

21. The axes being inclined at an angle $\omega$, a pair of lines which form a harmonic pencil with the imaginary pair $x^2+2xy\cos\omega+y^2=0$ are at right angles.

**273. Harmonic property of the quadrilateral.**

**In a quadrilateral $ABCD$ $BC$, $AD$ meet in $G$; $BA$, $CD$ in $E$; and $BD$, $AC$ in $F$. To show that the pairs of lines $GE$, $GF$ and $GA$, $GB$ are harmonic.**

Take $GBC$ and $GAD$ for axes; then, if $GA = \alpha$, $GB = \beta$,

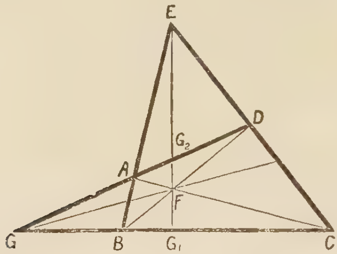



Fig. 96.

$GC = \gamma$, $GD = \delta$, we can write down the equations of lines as follows:—

$$\left.\begin{array}{l} AB \text{ is } \dfrac{x}{\beta} + \dfrac{y}{\alpha} = 1 \\ CD \text{ is } \dfrac{x}{\gamma} + \dfrac{y}{\delta} = 1 \end{array}\right\}, \qquad \left.\begin{array}{l} AC \text{ is } \dfrac{x}{\gamma} + \dfrac{y}{\alpha} = 1 \\ BD \text{ is } \dfrac{x}{\beta} + \dfrac{y}{\delta} = 1 \end{array}\right\}.$$

From the first two, we infer that, since $GE$ passes through the origin, its equation is

$$\left(\frac{x}{\beta} + \frac{y}{\alpha} - 1\right) - \left(\frac{x}{\gamma} + \frac{y}{\delta} - 1\right) = 0,$$

or
$$x\left(\frac{1}{\beta} - \frac{1}{\gamma}\right) + y\left(\frac{1}{\alpha} - \frac{1}{\delta}\right) = 0.$$

Similarly, from the last two we see that

$$GF \text{ is } \quad x\left(\frac{1}{\beta} - \frac{1}{\gamma}\right) - y\left(\frac{1}{\alpha} - \frac{1}{\delta}\right) = 0.$$

In the joint equation of $GE$ and $GF$ there is clearly no term in $xy$.

Therefore the lines $GE$, $GF$ form a harmonic pencil with the axes (§ 272, Cor. I.).

As an exercise, the reader should establish the following:—

COR. I. Similarly, by taking $EAB$, $EDC$ as axes, we can show that the pairs of lines $EF$, $EG$ and $EAB$, $EDC$ are harmonic.

COR. II. By taking $F$ as origin and $BFD$, $AFC$ as axes, we can show that the pairs $FE$, $FG$ and $FA$, $FB$ are harmonic.

**274. Complete quadrilateral and its properties.**

The whole figure is called a **complete quadrilateral.** To draw such a figure with ease, we first draw the lines $EAB$, $EDC$ meeting, and then the pair $GBC$, $GAD$ meeting. The reader will do well to notice this, because, if he

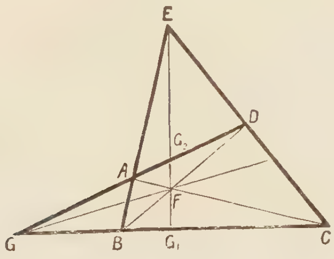



Fig. 97.

begins by taking the points $A$, $B$, $C$, $D$, the various lines frequently are so nearly parallel that the figure becomes too large.

It should also be noticed that the three points $E$, $F$, $G$ are, in reality, symmetrical with regard to the quadrilateral $ABCD$, for a point such as $E$, $F$, or $G$ is where the line joining *any* two of the points $A$, $B$, $C$, $D$ meets the line joining the other two.

A harmonic pencil of a complete quadrilateral is formed by the four lines drawn from *any* one of the three points of intersection $E$, $F$, $G$

(1) and (2) to the other two points of intersection;
(3) and (4) to the corners $A, B, C, D$ of the original quadrilateral.

275. The results of § 273 lead at once to a very interesting construction for finding the point harmonically conjugate to a point $G_1$ with respect to two given points $C$ and $D$.

For take any point $G$ and join $GC$, $GD$, $GG_1$. Then through $C$ draw any line $CA$ meeting $GG_1$ in $F$ and $GD$ in $A$. Join $DF$, and produce

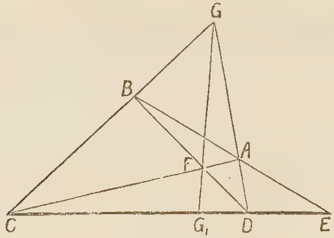



Fig. 98.

it to meet $GC$ in $B$. Finally, join $BA$, and produce it to meet $CD$ in $E$. Then $E$ is the point required, as follows at once from the above.

The foregoing construction may appear long and complicated to the reader, who will probably think that some such process as bisecting $CD$ in $O$ and then making $OG_1 \cdot OE = OD^2$ is more to the point. In reality, the construction we have given is probably the most important in the whole of geometry. This arises from the fact that it is effected by processes requiring the use of the *ruler only*. It may interest the student to analyze the Euclidean constructions, and see for himself that they all require the use of compasses as well.

A simpler construction is given in Ex. 14, p. 291.

**Exercise.**

22. In the complete quadrilateral of Fig. 97, if $EF$ produced meets $BC$ in $G_1$, show that the pairs of points $B$, $C$; $G$, $G_1$ are harmonic.

Hence, having proved that the pencil $E$ $(BFCG)$ is harmonic, deduce at once that $F$ $(GCG_1B)$ is; that is that $F$ $(GCEB)$ is harmonic.

[This all follows from the fact that a line meets a harmonic pencil in a harmonic range.]

276. **Two conjugate points with respect to a conic are harmonically conjugate to the points in which the line joining them meets the conic.**

In other words, if $A$ and $B$ be two points such that the polar of each passes through the other, and $AB$ meets the conic in $P$ and $Q$, then the pairs of points $A$, $B$ and $P$, $Q$ are harmonic.

Suppose that the conic is

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

that $A$ is $(x_1, y_1)$, and $B$ $(x_2, y_2)$. Then the polar of $A$ is

$$ax_1x+h(x_1y+xy_1)+by_1y+g(x+x_1)+f(y+y_1)+c=0;$$

therefore, since $B$ $(x_2, y_2)$ lies on it, we have

$$ax_1x_2+h(x_1y_2+x_2y_1)+by_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0.$$

Now the ratios in which the line $AB$ is divided by the conic are given by the well known ratio quadratic

$$k^2S_2+2klT_{12}+l^2S_1=0 \qquad (\S\ 136)$$

where

$$T_{12}=ax_1x_2+h(x_1y_2+x_2y_1)+by_1y_2+g(x_1+x_2)+f(y_1+y_2)+c,$$
$$S_1=ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c,$$
and
$$S_2=ax_2^2+2hx_2y_2+by_2^2+2gx_2+2fy_2+c.$$

Hence the coefficient of $kl$ is zero, and therefore the two ratios are equal and opposite, that is $P$ and $Q$ divide $AB$ internally and externally in the same ratio. This establishes the result.

277. **Alternative proof.**

Let $O$ and $A$ (Fig. 99) be the two points; take $O$ for origin, $OA$ for axis of $x$, and suppose the equation of the conic is

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

If the conic meets $OX$ in $P$ and $Q$, then, to find the lengths of $OP$, $OQ$ we must make $y$ zero in the above equation. Hence $OP$, $OQ$ are the roots of the equation

$$ax^2+2gx+c=0,$$

so that, by the theory of equations, we have

$$\frac{1}{OP}+\frac{1}{OQ}=-\frac{2g}{c}.$$

But the polar of $O$ (0, 0) is $gx+fy+c=0$, and, as this passes through $A$, by hypothesis, $OA$ is given by

$$gx+c=0;$$

$$\therefore \quad \frac{1}{OA}=-\frac{g}{c}.$$

Comparing the two results, we see at once that

$$\frac{2}{OA}=\frac{1}{OP}+\frac{1}{OQ},$$

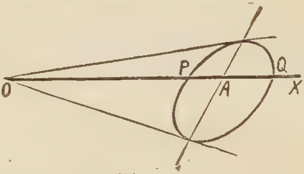


Fig. 99.

which shows at once that $O$ and $A$ are harmonic conjugates to $P$ and $Q$. (§ 265.)

This result may also be stated as follows:—

*A line through any point is divided harmonically by the point, the curve, and the polar of the point.*

Cor. If a line be drawn through a fixed point $O$ to meet a conic in $P$ and $Q$, and a point $R$ be taken on the line such that $OR$ is the harmonic mean between $OP$ and $OQ$, then as the line $OPQ$ turns round $O$ the locus of $R$ is a straight line.

For the point in which the line $OPQ$ meets the polar of $O$ divides it in the given manner, by the theorem above, and hence $R$ is always on the polar of $O$.

**Exercises.**

23. Verify that the points (1, 1) $(\frac{4}{3}, \frac{4}{3})$ are conjugate points with respect to the conic $x^2+xy+y^2=4$. Find the quadratic for the ratio in which the line joining them is divided by the curve, and hence verify that the section is harmonic.

24. Prove that, if the line $AB$ is divided harmonically by the curve, then the polar of $A$ passes through $B$.

25. $A$, $B$, $C$, $D$ are four points on a conic; $AB$, $CD$ meet in $E$, $BC$, $AD$ in $G$, and $AC$, $BD$ in $F$. Then $EF$ meets $AD$, $BC$ in $G_1$. $G_2$ respectively. Show, from the properties of the complete quadrilateral, that the pairs of points $G$, $G_1$; $D$, $A$ and the pairs $G$, $G_2$; $C$. $B$ are harmonic. Deduce that the polar of $G$ passes through $G_1$ and $G_2$ and is therefore the line $EF$.

26. Show that in Ex. 25 the triangle $EFG$ is such that each side is the polar of the opposite vertex with respect to the conic.

(Such a triangle is said to be **self-conjugate** with respect to the conic.)

**278. Two conjugate lines with respect to a conic form a harmonic pencil with the tangents drawn from their point of intersection to the conic.**

Suppose that $OA$, $OB$ are two lines conjugate with respect to the conic, that is, such that the pole of each lies on the other. Then, if $OT_1$, $OT_2$ be the tangents from $O$, we have to show that the two pairs of lines $OA$, $OB$ and $OT_1$, $OT_2$ are harmonic.

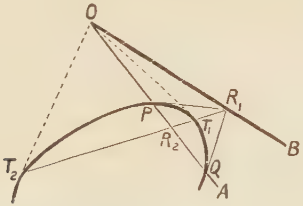



Fig. 100.

Let the line $OA$ meet the conic in $P$ and $Q$; then, by hypothesis, the pole of $OA$ lies on $OB$. But the pole of $OA$ is the point of intersection of the tangents at $P$ and $Q$.

Therefore these tangents meet on $OB$, say at the point $R_1$.

Now $PQ$, the polar of $R_1$, passes through $O$.

Therefore $T_1T_2$, the polar of $O$, passes through $R_1$.

Let $T_1T_2$ meet $PQ$ in $R_2$. Then, by § 276, $R_1$ and $R_2$ are conjugate to $T_1$ and $T_2$.

Therefore $OR_1$, $OR_2$ form a harmonic pencil with $OT_1$, $OT_2$ (§ 270), *i.e.* the pairs of lines $OA$, $OB$; $OT_1$, $OT_2$ are harmonic.

279. **Alternative proof.**—We shall now give a purely analytical proof of the last theorem, on account of the importance of the theorem and the instructiveness of the method.

Suppose the conjugate lines are the axes of coordinates, so that $O$ is now the origin.

As the axes are clearly not, in general, conjugate lines of

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

some condition will have to be satisfied among the coefficients. To find it we must find under what circumstances the pole of $x = 0$ lies on $y = 0$.

Now the polar of $(x', y')$ is

$$x(ax' + hy' + g) + y(hx' + by' + f) + gx' + fy' + c = 0,$$

so that, for the pole of $x = 0$, we have the two equations

$$hx' + by' + f = 0, \quad gx' + fy' + c = 0.$$

If this point lie on $y = 0$, we have

$$hx' + f = 0, \quad gx' + c = 0;$$

hence

$$fg - ch = 0.$$

Now the tangents from $O$ to the curve are

$$(gx + fy + c)^2 = c(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) \qquad (\S\ 138)$$

or

$$x^2(ac - g^2) - 2xy(fg - ch) + y^2(bc - f^2) = 0,$$

and the term in $xy$ is zero in virtue of the above.

Consequently, from § 272, it follows at once that the pair of tangents and the axes are harmonic.

280. As an illustration of the foregoing theorem, consider the conjugate diameters of a central conic.

Conjugate lines through the centre are conjugate diameters, for the pole of a diameter is the point of intersection of the tangents at its ends. These are both parallel to the conjugate diameter. and therefore meet in the point at infinity on the conjugate diameter (§ 200); therefore the pole of a diameter is on the conjugate diameter. Hence, as stated, conjugate diameters are conjugate lines through the centre. The tangents from the centre are the asymptotes, and therefore

*A pair of conjugate diameters form a harmonic pencil with the asymptotes.*

## MISCELLANEOUS EXERCISES ON CHAP. XX.

27. Points are taken on a line distant

$$a,\ a + \frac{2xy}{x+y}; \quad a + x,\ a + y$$

from a given point on the line. Show that the first pair are harmonic to the second.

28. Show that the points where the internal and external bisectors of the vertical angle of a triangle meet the base are harmonically conjugate with respect to the ends of the base.

29. Write down the condition that the pairs of lines

$$x^2 - y^2 = 0, \quad ax^2 + 2hxy + by^2 = 0$$

may form a harmonic pencil.

30. Show that there is only one pair of lines harmonic with respect to each of two given pairs.

31. If the axes be rectangular, and $Ax^2+2Hxy+By^2=0$ be the bisectors of the angles between the lines $ax^2+2hxy+by^2=0$, then show that (i.) $A+B=0$, (ii.) $Ab+Ba-2Hh=0$.

By solving these equations for $A$ and $B$ deduce the equations for the bisectors.

32. Show that two points on the axis of $x$ at distances $\alpha$, $\beta$ from the origin are harmonic to two whose distances are given by $ax^2+2bx+c=0$ if $a\alpha\beta+b(\alpha+\beta)+c=0$.

[Form the quadratic of which $\alpha$ and $\beta$ are the roots.]

33. If the lines $y=m_1x$, $y=m_2x$ and $ax^2+2hxy+by^2=0$ are harmonic, then show that

$$am_1m_2+h(m_1+m_2)+b=0.$$

34. Prove that the diagonals of a parallelogram, together with the lines joining the middle points of opposite sides, form a harmonic pencil.

35. Given a harmonic range $ABCD$ and any point $O$, prove that a straight line drawn through $B$ parallel to $OD$ is met by $OA$ and $OC$ in points equidistant from $B$.

Show also that, if $abcd$ is another harmonic range such that the straight lines $Aa$, $Bb$, $Cc$ meet in a point, then the straight line $Dd$ passes through the same point.

36. The two pairs of lines $y=m_1x$, $y=m_2x$; $y=m_3x$, $y=m_4x$ are harmonic, if only $\dfrac{m_1-m_3}{m_2-m_3}\cdot\dfrac{m_2-m_4}{m_1-m_4}=-1.$

[Make the points in which they meet $x=1$ harmonic.]

37. If the two pairs of lines $y=m_1x$, $y=m_2x$; $y=m_3x$, $y=m_4x$ are harmonic, then so also are the two pairs

$$y=\frac{a+bm_1}{c+dm_1}x,\ \ y=\frac{a+bm_2}{c+dm_2}x;\quad y=\frac{a+bm_3}{c+dm_3}x,\ \ y=\frac{a+bm_4}{c+dm_4}x,$$

where $a$, $b$, $c$, $d$ are any constants.

38. If four diameters of an ellipse form a harmonic pencil, then so also do the four diameters respectively conjugate to them.

39. If a triangle be inscribed in a circle, and a diameter of the circle be drawn at right angles to one of the sides, the remaining two sides, produced if necessary, will divide this diameter harmonically.

40. Through the origin of coordinates $O$ straight lines are drawn, any one of which $OPQS$ cuts the fixed line $lx+my+n=0$ at $P$, and the fixed conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ at $Q$ and $S$. Prove that the locus of the point $R$, which is such that $P$, $Q$, $R$, $S$ are four harmonic points, may be expressed thus:—

$$n(ax^2+2hxy+by^2+2gx+2fy+c)=(lx+my+n)(gx+fy+c).$$

## EXAMINATION PAPER VI.

1. If $$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$
and $$S' = a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c',$$
show that the equation of any conic which passes through the four points common to $S = 0$ and $S' = 0$ is of the form $S + \lambda S' = 0$.

The axes being supposed rectangular, show that the condition that the four common points should lie on a circle is
$$h : h' = a - b : a' - b'.$$

2. Show that the conic represented by
$$ax^2 + by^2 + 2hxy + 2f\cos^2\theta \,.\, y + 2g\sin^2\theta \,.\, x + c = 0,$$
where $\theta$ is a variable angle, always passes through two fixed points.

3. Find the equation of the circle circumscribing the triangle formed by $x = 1$, $y = 2$, $x - y = 0$.

4. Interpret the equation $\alpha\beta = \gamma^2$ where $\alpha = 0$, $\beta = 0$, $\gamma = 0$ are the equations of three straight lines. Show that the equation $xy = \lambda^2(x/a + y/b - 1)^2$ represents a conic section touching the axes of coordinates.

5. Find the nature and position of the curve touched by the family of straight lines represented by the equation
$$\mu^2(2x + 3y + 4a) + \mu(3x + 4y + 5a) + 6a = 0.$$

6. The axes of an ellipse are given in position. Find its envelope if the product of its axes is constant.

7. Define the term *evolute*; and find the equation to the evolute of a parabola.

8. On the straight line $OX$ there are situated four harmonic points $P$, $Q$, $R$, $S$, of which $P$ and $R$, $Q$ and $S$ are conjugate pairs, all the four points being on one side of $O$. Prove that
$$(OP + OR)(OQ + OS) = 2OP \,.\, OR + 2OQ \,.\, OS.$$

9. If $A$, $B$, $C$, $D$ are four points in a plane; and, if $AB$, $DC$ intersect at $G$; $BC$, $AD$ at $F$; and $AC$, $BD$ at $E$, then prove that the straight lines $E\,(BGCF)$ form a harmonic pencil.

Having given three points in a straight line, find, by a construction which can be made by the ruler only, the fourth harmonic of one of the points for the other two.

10. In a conic, prove that a straight line, drawn through any point, is cut harmonically by the point, the curve, and the polar of the point.

What does the preceding theorem become if the right line is parallel to an asymptote of a hyperbola or to a diameter of a parabola, or if the point is at the centre of the conic?

# CHAPTER XXI.

## CROSS RATIOS.

281. Definition.—When four points $A$, $C$, $B$, $D$ are taken on a straight line (Fig. 101) the ratio

$$\frac{AC.BD}{AD.BC}$$

is called the **anharmonic ratio** or **cross ratio** of the range $ACBD$, and is written $(ACBD)$. The usual convention of signs is to be observed, *e.g.* $AC = -CA$.

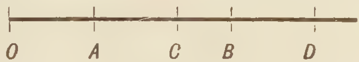



Fig. 101.

The method of writing down the fraction should be observed: in the numerator the points occur in the given order, while in the denominator the second and fourth points are interchanged.

In the same way if we have four concurrent lines $OA$, $OC$, $OB$, $OD$ (Fig. 102), the ratio

$$\frac{\sin AOC . \sin BOD}{\sin AOD . \sin BOC}$$

is called the **anharmonic ratio** or **cross ratio** of the pencil of rays $OA$, $OC$, $OB$, $OD$, and is written $O(ACBD)$.

282. **Cross ratios are projective.** — Let $p$ be the perpendicular from $O$ on $ACBD$. Then we have

$$AC \cdot p = 2\triangle AOC$$
$$= OA \cdot OC \sin AOC$$
or $$AC = \frac{OA \cdot OC \sin AOC}{p}.$$

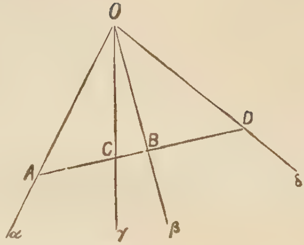


Fig. 102.

Similar values can be found for $BD$, $AD$, $BC$.

Hence the cross ratio $(ACBD)$

$$= \frac{OA \cdot OC \sin AOC \times OB \cdot OD \sin BOD}{OA \cdot OD \sin AOD \times OB \cdot OC \sin BOC}$$
$$= \frac{\sin AOC \cdot \sin BOD}{\sin AOD \cdot \sin BOC} = O(ACBD).$$

Let now any other transversal cut the pencil in $A'C'B'D'$.

Then $$(A'C'B'D') = O(A'C'B'D') = O(ACBD)$$
$$= (ACBD).$$

It follows that the value of a cross ratio is unaltered by projection (see definition of projection in § 270 : this kind of projection is called *conical* projection).

Now it is proved in Geometrical Conics that by means of conical projection a circle can be projected into any kind of conic—parabola, ellipse, or hyperbola—and *vice versa* any conic can be projected into a circle. It follows that any property in cross ratios which can be proved for a circle is also true for any conic section.

The value of a cross ratio may be expressed thus. Let $A, C, B, D$ be at distances $a, c, b, d$ from a fixed origin $O$ on the line $ACBD$. Then clearly (Fig. 101)

$$\frac{AC \cdot BD}{AD \cdot BC} = \frac{(c-a)\ (d-b)}{(d-a)\ (c-b)}.$$

Note that in Fig 101, $BC$ will be negative.

The particular case when the cross ratio $= -1$ is that of a harmonic ratio; see § 263, Cor.

283. **Different cross ratios with the same four points.** Given four points on a line, if they are taken in different orders, different cross ratios are of course obtained. There are $4P_4$ or 24 orders, but these do not all give different values for the cross ratio. It can be verified at once indeed on writing down the values that

$$(ABCD) = (BADC) = (CDAB) = (DCBA),$$

and similarly for other arrangements. There will thus be only $24 \div 4$ or 6 different values, and these are found to be related to each other.

LEMMA.—It is easily verified, attention being paid to sign, that

$$AC.BD + AB.DC + AD.CB = 0.$$

For the expression equals

$$\begin{aligned}&(AD + DC)\,BD + (AD + DB)\,DC + AD.CB \\ &= AD\,(BD + DC + CB) + DC\,(BD + DB) \\ &= 0\end{aligned}$$

since $$BD + DC + CB = BD + DB = 0$$

We have now

$$(ADBC) = \frac{AD.BC}{AC.BD} = \frac{1}{(ACBD)} \quad\text{.................}\quad \text{(I.)}.$$

$$(ABCD) = \frac{AB.CD}{AD.CB} = \frac{AB.DC}{AD.BC} \quad\text{.................}\quad \text{(II.)},$$

$$\therefore \quad (ACBD) + (ABCD) = \frac{AC.BD + AB.DC}{AD.BC} = 1$$

by the lemma just proved, or

$$(ABCD) = 1 - (ACBD),$$

and so for other arrangements. The reader may verify as an exercise that if $(ACBD)$ is $\lambda$, the other five ratios are

$$\frac{1}{\lambda},\quad 1-\lambda,\quad \frac{1}{1-\lambda},\quad \frac{\lambda}{\lambda-1},\quad \frac{\lambda-1}{\lambda}.$$

284. **Examples.**—If $A$, $B$, $C$, $D$ are fixed points on a circle and $O$ any other point on the circle, the cross-ratio $O\,(ACBD)$ is independent of the position of $O$. This is immediately obvious, since the values of the angles $AOC$, $COB$, $BOD$, $DOA$ are independent of the position of $O$ (Euc. iii. 21).

Projecting the circle into a conic, we have the important result that **the pencil formed by joining four fixed points on the curve to any other point on it has a constant cross-ratio.**

Again:—

**If four fixed tangents be drawn to a conic they will intercept on any fifth tangent a range whose cross-ratio is constant.**

For suppose the tangents at the fixed points $A$, $B$, $C$, $D$ on a given circle cut the tangent at $P$ in the points $L$, $M$, $N$, $R$. If $O$ is the centre of the circle,

$$\angle LOM = \angle LOP - \angle MOP = \tfrac{1}{2}\angle AOP - \tfrac{1}{2}\angle BOP = \tfrac{1}{2}\angle AOB,$$

and is therefore constant. The angles of the pencil $O\,(LMNR)$ are thus constant, and so therefore is the cross-ratio of the pencil and therefore also of the range. The theorem is extended to the conic by projection.

285. **If two pencils of equal cross-ratio have a common ray, the points of intersection of the other three corresponding rays are collinear.**

For let $PP'$ be the common ray, the pencils being $P\,(P'ABC)$ and $P'\,(PABC)$: so that $A$, $B$, $C$ are the points of intersection. If possible, let $AC$ cut $PB$ and $P'B$ in two different points $B'$ and $B''$; and let $AC$ meet $PP'$ in $D$. Then, since the pencils have equal cross-ratio,

$$(DAB'C) = (DAB''C),$$

which is impossible unless $B'$ and $B''$ coincide.

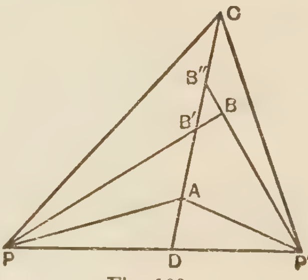



Fig. 103.

286. **Pascal's Theorem.—If a hexagon (of any shape) be inscribed in a circle or a conic, the points of intersection of opposite sides lie on a straight line.**

Let $ABCDEF$ be the hexagon. Let $AB$, $DE$ cut in $L$; $BC$, $EF$ in $M$; $CD$, $FA$ in $N$. Let $BC$, $DE$ cut at $P$; and $CD$, $EF$ at $Q$. Join $LM$, $MN$.

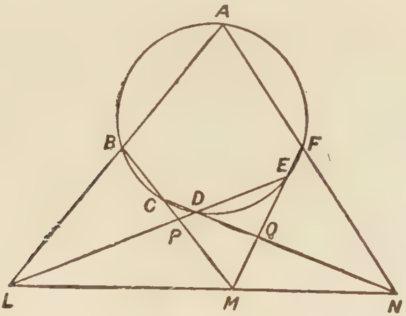



Fig. 104.

The anharmonic ratio

$$M(LCDE) = (LPDE) = B(LPDE) = B(ACDE).$$

Similarly

$$M(NCDE) = (NCDQ) = F(NCDQ) = F(ACDE).$$

But $$B(ACDE) = F(ACDE);$$

$$\therefore \quad M(NCDE) = M(LCDE),$$

and since three rays $MC$, $MD$, $ME$ are identical, so must $ML$ and $MN$ be. Hence $LMN$ are collinear.

From Pascal's theorem we can deduce **a method of drawing a conic through five given points,** using a ruler only. For let $A$, $B$, $C$, $D$, $E$ be the given points. Then, in Fig. 104, if any line through $E$ cut the conic again in $F$, the intersections of $AB$, $DE$ and $BC$, $EF$ and $CD$, $FA$ are collinear.

The construction for $F$ therefore is; — let $AB$ and $DE$ meet in $L$, and $BC$ cut the line through $E$ in $M$. Let $CD$ cut $LM$ in $N$. Then $NA$ will cut the line drawn through $E$ in $F$.

Since any number of lines can be drawn through $E$, we can thus obtain any number of points on the curve, and then draw the conic through them.

287. **Involution. — If on a line $OX$ we have three pairs of points $A, A'$; $B, B'$; $C, C'$ such that**

$$OA.OA' = OB.OB' = OC.OC' = k^2;$$

**then the cross-ratio of any four of the six points is equal to that of their four conjugates.**

For draw $OY$ perpendicular to $OX$ such that $OY^2 = k^2$ (Fig. 105). Then

$$OY^2 = OA.OA',$$

and hence the circle $AYA'$ touches $OY$. Hence

$$\angle OYA = \angle OA'Y.$$

In like manner

$$\angle OYB = \angle OB'Y;$$

and hence, by subtraction,

$$\angle AYB = A'YB'.$$

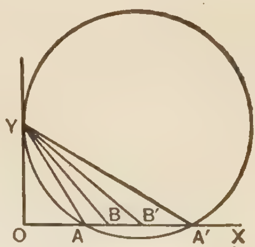


Fig. 105.

Similarly $BYC = B'YC'$, &c.; and thus the angles of a pencil $Y(ABCC')$ are respectively equal to the angles of pencil $Y(A'B'C'C)$. Hence the cross-ratios of these pencils and therefore also of the ranges are equal.

The points $A$, $A$; $B$, $B'$... it will be observed are inverses with respect to a circle whose centre is $O$.

The points $A$, $A'$, ... are said to form a system in **involution** of which $O$ is the centre. If $F$, $F'$ be taken on $OX$ (on either side of $O$), so that $OF^2 = OF'^2 = k^2$, then $F$, $F'$ are called the **foci** of the involution.

## Exercises.

1. If $(ABCD) = -1$ prove that $(ACBD) = 2$ and $(ACDB) = \frac{1}{2}$.

2. If the pencil $O(ACBD) = 1$, prove that two rays of the pencil coincide. (Use § 282.)

3. If circles described on $AB$, $CD$, as diameters intersect at an angle $\theta$, show that the values of the 6 cross ratios of the ranges formed by the points $A$, $B$, $C$, $D$, are

$$-\tan^2\frac{\theta}{2},\quad \sec^2\frac{\theta}{2},\quad \sin^2\frac{\theta}{2},\quad -\cot^2\frac{\theta}{2},\quad \cos^2\frac{\theta}{2},\quad \operatorname{cosec}^2\frac{\theta}{2}.$$

4. If $A$, $B$, $C$, and $A'$, $B'$, $C'$ are two triads of points on two straight lines $OABC$, $OA'B'C'$ meeting in $O$ such that $(OABC) = (OA'B'C')$; prove that $AA'$, $BB'$, $CC'$ are concurrent.

5. On the ellipse $x^2 + 4y^2 = 100$ are four points $A$, $B$, $C$, $D$ whose coordinates are $(10, 0)$, $(0, -5)$, $(-8, -3)$, $(-10, 0)$, and $P$ is any point on the ellipse. Find the cross ratio of the pencil $P(ABCD)$.

6. Given four points $A$, $B$, $B'$, $A'$ in one straight line, find the locus of a point $P$, such that the angles $APB$, $B'PA'$ are equal.

# PROBLEM PAPERS.

## Problem Paper 1.

1. Find the coordinates of a point such that the line joining it to the point $(p, q)$ is bisected at right angles by the line $lx+my+n=0$.

2 A point moves so that the sum of the squares of its distances from two given sides of an equilateral triangle is constant and equal to $2c^2$ Show that the locus is an ellipse, and find the eccentricity and the position of the foci.

3. Prove that the tangent at any point of the ellipse $x^2/a^2+y^2/b^2=1$ meets the ellipse $x^2/a^2+y^2/b^2=k$ (where $k>1$) in two points equidistant from the point of contact.

4. Normals are drawn to an ellipse from a given point, inclined at angles $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ to the major axis. Show that, if $\theta$ be the inclination to the axis of the line joining the given point to the centre of the ellipse, $\tan\theta_1+\tan\theta_2+\tan\theta_3+\tan\theta_4 = 2\tan\theta$.

5. A circle is described with its centre at the foot of the directrix of a parabola, and its diameter equal to the latus rectum. Prove that the locus of the pole of any tangent to the parabola with regard to this circle is an equilateral hyperbola.

6. Show that the asymptotes of the curve $x^2+2xy-y^2-2x-2y=0$ are

$$x+y-1 = \pm\sqrt{2}.y,$$

and trace the curve.

7. Determine the locus of the centre of a variable circle which touches a fixed circle and a fixed line.

8. If $a'$, $b'$ be the lengths of $CP$, $CD$, conjugate diameters of an ellipse, and $\theta_1$, $\theta_2$ their inclinations to the axis of $x$, then

$$a'^2\sin 2\theta_1+b'^2\sin 2\theta_2 = 0.$$

9. If from any point on a given straight line tangents be drawn to a given conic, prove that the sum or difference of the reciprocals of the perpendiculars let fall on the line from the points of contact of the tangents is constant.

10. Show that the equation

$$\frac{1}{x+y-a}+\frac{1}{x-y+a}+\frac{1}{y-x+a}=0$$

represents a parabola. Find its focus and directrix.

## Problem Paper 2.

1. Investigate the locus of a point such that the sum of the squares of its distances from two of the vertices of a given triangle may be equal to the square of its distance from the third vertex.

2. Prove that, if the straight line $\lambda x + \mu y + \nu = 0$ touches the parabola

$$y^2 - 4px + 4pq = 0,$$

then must

$$\lambda^2 q + \lambda\nu - p\mu^2 = 0.$$

3. Pairs of tangents are drawn to the ellipse $x^2/a^2 + y^2/b^2 = 1$ from points lying on the ellipse

$$x^2/a^4 + y^2/b^4 = 1/a^2 + 1/b^2.$$

Show that their points of contact will subtend a right angle at the centre.

4. If $(x_1, y_1)$ and $(x_2, y_2)$ are two points on the ellipse $x^2/a^2 + y^2/b^2 = 1$, the tangents at which meet in $(x, y)$ and the normals in $(\xi, \eta)$, prove that

$$a^2\xi = e^2 x x_1 x_2 \quad \text{and} \quad b^2\eta = e^2 y y_1 y_2,$$

where $e$ is the eccentricity.

5. Find the coordinates of the centre, and the area, of the ellipse whose equation is $x^2 + 3xy + 4y^2 - 28x - 56y + 196 = 0$.

6. Show that the axes are tangents to the above curve, and that the chord of contact is bisected by the line joining the centre to the origin.

7. From any point on the normal to $x^2/a^2 + y^2/b^2 = 1$ at the point whose eccentric angle is $\alpha$ two other normals are drawn to the ellipse. Prove that the locus of the point of intersection of the corresponding tangents is the hyperbola $bx \sin\alpha + ay \cos\alpha + xy = 0$.

8. A system of conics have the same focus and directrix. Find the locus of points the tangents at which are parallel to a given line.

9. If the sum of the ordinates of two points on the ellipse $x^2/a^2 + y^2/b^2 = 1$ be $b$, show that the locus of the pole of the chord which joins them is $b^2x^2 + a^2y^2 = 2a^2by$.

10. A point $P$ is taken on the conic whose equation is

$$x^2/(a^2 + \lambda) + y^2/(b + \lambda) = 1$$

such that the normal at $P$ may pass through a fixed point $(h, k)$. Show that $P$ lies on the curve given by

$$\frac{x}{y-k} + \frac{y}{x-h} = \frac{a^2 - b^2}{hy - kx}.$$

**Problem Paper 3.**

1. Determine the conditions that the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

may represent one straight line.

2. Find the radius, and the polar coordinates of the centre, of the circle $$r^2 - 3r(\cos\theta + \sqrt{3}\sin\theta) + 5 = 0.$$

3. Determine the area of the ellipse whose equation referred to rectangular axes is

$$(A^2+1)x^2 + 2(A+C)xy + (C^2+1)y^2 = F.$$

4. Find a common tangent to the ellipses

$$x^2/(a^2+b^2) + y^2/b^2 = 1$$

and $$x^2/a^2 + y^2/(a^2+b^2) = 1,$$

and show that it is parallel to a diagonal of the parallelogram whose sides are their directrices.

5. Prove that, if $a^2 > 8b^2$, a point can be found the two tangents from which to $y^2 - 4ax = 0$ are normals to $x^2 - 4by = 0$.

6. Trace the curves :—

(i.) $y^2 + 2xy + x^2 + y - 3x + 1 = 0$;

(ii.) $5x^2 - 6xy + y^2 - 2y - 2x + 1 = 0$.

7. Find the area of the triangle subtended at the centre of the ellipse $x^2/a^2 + y^2/b^2 = 1$ by the chord joining the points whose eccentric angles are $\alpha$ and $\beta$.

8. An ellipse with a given point $O$ as centre passes through three given points $A$, $B$, $C$. Show that the area of the ellipse

$$= \frac{4\pi LMN}{\sqrt{(L+M+N)(M+N-L)(N+L-M)(L+M-N)}},$$

where $L$, $M$, $N$ denote the areas of the triangles $BOC$, $COA$, $AOB$.

9. Discuss and draw the curve

$$2x^2 + xy + y^2 = 14x,$$

and show how the conic $2xy = a(x+y)$ is situated with respect to the conic $2xy = a^2$.

10. A parabola whose axis passes through the centre of a given ellipse, and whose latus rectum is $\frac{3}{5}$ of its minor axis, touches the ellipse at the extremity of the minor axis. Find the equation of the parabola, and that of the chord of intersection of the two curves.

### Problem Paper 4.

1. Form the equation representing the straight lines joining the origin to the points of intersection of the straight line

$$x/a + y/b = 1$$

and the circle $\qquad 5(x^2 + y^2 + bx + ay) = 9ab,$

and find the condition that they should include a right angle.

2. The equations to two circles are

$$(x-a)^2 + (y-b)^2 = c^2, \quad (x-b)^2 + (y-a)^2 = c^2.$$

Show that the condition of their touching each other is $2c^2 = (a-b)^2$.

3. $PQ$ is a focal chord of an ellipse, of which $BB'$ is the minor axis. Show that the locus of the intersection of $BP$ and $B'Q$ is a hyperbola.

4. Two tangents $TP$, $TQ$ are drawn to the ellipse from the point $T$, whose coordinates are $h$, $k$. Show that the area of the triangle $TPQ$ is

$$ab\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right)^{\frac{3}{2}} \Big/ \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right).$$

5. If two normals to a parabola intersect at right angles, show that the portion of the one intercepted by the curve is cut by the other into two parts which have to each other the ratio $3:1$.

6. Trace the curve

$$8x^2 - 2xy - 15y^2 + 6x + 46y - 35 = 0.$$

7. Show that the locus of a point the two tangents from which to the ellipse $x^2/a^2 + y^2/b^2 = 1$ intersect at an angle equal to the difference between the eccentric angles of their points of contact is one or other of the two curves

$$x^2/a + y^2/b = a + b, \quad x^2/a - y^2/b = a - b.$$

8. Show that the locus of the middle point of the chord joining the ends of conjugate diameters of $x^2/a^2 + y^2/b^2 = 1$ is the ellipse

$$x^2/a^2 + y^2/b^2 = \tfrac{1}{2}.$$

9. Two points $P$ and $Q$ are taken on an ellipse so that the pole of the normal at $P$ lies on the normal at $Q$. Prove that the relation between $P$ and $Q$ is reciprocal; and show that, if $K$ be the point of intersection of the normals at $P$ and $Q$, the line joining $K$ to the pole of the chord $PQ$ is at right angles to the polar of $K$.

10. Being given, of a triangle, its vertical angle $\alpha$ in position and the sum of $m$ times one side and $n$ times the other $= c$, find the locus of the centre of the circumscribed circle.

## Problem Paper 5.

1. If
$$1 + hx + ax^2 + bxy + cy^2 = 0$$
represents two real straight lines, prove that $c$ must be negative, and find $h$ in terms of $a$, $b$, $c$.

2. Find the circle passing through $(4d, 0)$, $(d, 0)$ and touching $\theta = \alpha$.

3. Any two perpendicular focal chords of the parabola $y^2 = 4ax$ are drawn, and from the middle point of each chord a line is drawn at right angles to that chord. Prove that the locus of their point of intersection is
$$y^2 = 2a(x - 5a).$$

4. A chord of an ellipse subtends a right angle at the centre. Show that the chord always touches a fixed circle.

5. The normals at the points $P$, $D$, extremities of conjugate diameters, meet the major axis in $G$, $G'$. Prove that
$$PG^2 + DG'^2 = \frac{b^2}{a^2}(a^2 + b^2).$$

6. If the sum of the eccentric angles of the points of contact of two tangents to an ellipse is constant, what is the locus of their point of intersection?

7. If $\theta$ be the inclination to $SP$ of the tangent at $P$ to an ellipse, and $\phi$ of the tangent at $D$ to $SD$ the conjugate semi-diameter, then
$$\cot^2\theta + \cot^2\phi$$
is constant.

8. Find the equation of the circle which passes through the points whose coordinates are $(1, 0)$, $(2, 3)$ and $(3, -1)$.

Show that the length of the intercept which it cuts off from the line $x - y = 0$ is 3·39, approximately.

9. If a chord of a circle subtends a right angle at a given point, show that the chord envelops a conic.

10. Two parabolas
$$y^2 = 4a(x - l)$$
and
$$x^2 = 4a(y - l')$$
move so as always to touch one another, $l$ and $l'$ being both variable. Find the locus of their point of contact.

## Problem Paper 6.

1. If $ABC$ be a plane triangle such that $\tan A \tan \frac{1}{2}B = 2$, find the locus of $C$ if $AB$ be fixed.

2. Find the area of the triangle formed by the lines whose equations are, respectively,

$$r\cos\theta - c = 0,$$
$$r\cos(\theta - \tfrac{1}{6}\pi) - \tfrac{1}{2}c = 0,$$
$$r\cos\theta + r\sin\theta + 2c = 0.$$

3. Find the equation of the common tangent to the parabolas

$$y^2 = 4ax \text{ and } x^2 = 4by.$$

4. If the normal to an ellipse at the point $P$ meet the major axis in $G$, and $CY$ be the perpendicular from the centre $C$ on the tangent at $P$, $O$ the middle point of $CG$, and $B$ the extremity of the minor axis, prove that

$$OP = OY = OB.$$

5. Trace the conics $3x^2 \pm 2xy + 3y^2 = 8$;
and show that the foci of each lie on the other. Also show that the conics have four points through which they both pass.

6. Show that, if through the point $(\alpha, \beta)$ on a parabola two normals to the curve be drawn to meet it in the points $(\alpha', \beta')$, $(\alpha'', \beta'')$, then

$$\beta'\beta''(\alpha' - \alpha'') + \beta''\beta(\alpha'' - \alpha) + \beta\beta'(\alpha - \alpha') = 0.$$

7. Show that the locus of the point of intersection of tangents to

$$x^2/a^2 + y^2/b^2 = 1$$

at the ends of conjugate diameters is the ellipse

$$x^2/a^2 + y^2/b^2 = 2.$$

8. Show that $(ay - bx)^2 + k(x-a)(y-b) = 0$
represents two straight lines; also that they both touch the hyperbola

$$xy - \tfrac{1}{4}k = 0.$$

9. Show that the four lines given by the equation

$$x^4 + 7x^3y + 15x^2y^2 + 7xy^3 - 6y^4 = 0$$

form a harmonic pencil.

10. Being given the base of a triangle and the ratio of the rectangle of its other sides to the sum of their squares, find the locus of the vertex of the triangle.

## Problem Paper 7.

1. Find the equation of the perpendicular from the point $(h, k)$ on the line
$$ax+by+c=0;$$
and find the coordinates of the point of intersection of the lines.

2. Prove that the distance of the point of intersection of the lines in Q. 1 from the point $(h, k)$ is
$$\left\{(ak+bh+c)^2+(a+b)^2(h-k)^2\right\}^{\frac{1}{2}}/(a^2+b^2)^{\frac{1}{2}}.$$

3. A circle whose centre lies on the straight line which bisects the angle between the coordinate axes touches each of the straight lines
$$Ax+By+C=0, \quad Ax+By+D=0.$$
Find its equation.

4. $A$ is the vertex of a parabola and $P$, $Q$ are any two points on the curve. The diameters through $P$, $Q$ meet $AQ$, $AP$ respectively in $U$, $V$. Show that $UV$ is perpendicular to the axis of the parabola.

5. Prove that the two parabolas
$$y^2=8ax, \quad x^2=27ay$$
intersect at right angles and at the angle whose tangent is $\frac{9}{13}$.

6. The normal at $P$ meets the tangent at $Q$ on the minor axis. Prove that $PQ$ touches the hyperbola
$$\frac{x^2}{a^2(a^2-2b^2)}-\frac{y^2}{b^4}=\frac{1}{a^2-b^2}.$$

7. Find the locus of the middle points of all tangents drawn from the directrix to the parabola $y^2=4ax$.

8. If a variable point on a hyperbola be joined to two fixed points on the curve, prove that the two joining lines intercept a segment of constant length on either asymptote.

9. Prove that the locus of a point such that the normals to the parabola
$$y^2=4ax$$
at the points of intersection of the curve with the polar of the point intersect on the parabola is the curve
$$y^2(x+2a)+4a=0.$$

10. Prove that the envelope of chords of an ellipse which subtend a right angle at the origin is a conic whose focus is the origin and directrix the polar of the origin.

## Problem Paper 8.

1. If $$ax^2+2hxy+by^2+2gx+2fy+c=0$$
represent two straight lines equidistant from the origin, show that
$$f^4-g^4=c\,(bf^2-ag^2).$$

2. Two circles are drawn through the points $(a, 5a)$ and $(4a, a)$ and touching the axis of $y$. Prove that they intersect at an angle $\tan^{-1}\frac{40}{9}$.

3. On $QP_1$, $QP_2$, tangents to a parabola, perpendiculars $SY_1$, $SY_2$ are let fall from the focus. Show that $SY_1 . SY_2$ varies as $SQ$.

4. Prove that the angle between the tangents drawn from $(x, y)$ to the ellipse $$x^2/a^2+y^2/b^2-1=0$$
is $$\tan^{-1}\left\{2ab\sqrt{x^2/a^2+y^2/b^2-1}\,/\,(x^2+y^2-a^2-b^2)\right\}.$$

5. Prove that the area of the triangle formed by three normals to a parabola which do not pass through one point is
$$\tfrac{1}{2}a^2\,(m+m'+m'')^2\,(m-m')\,(m'-m'')\,(m''-m),$$
where $4a$ is the latus rectum of the parabola and $m$, $m'$, $m''$ are the tangents of the angles made by the normals with the axis of the curve.

6. The equation of a conic is
$$x^2-4xy+y^2+6x=6.$$
Show that it is a hyperbola; find its centre, the direction and length of its semi-axes, and the equation of the pair of asymptotes.

7. Prove that the chords which pass through a point $(x', y')$ within a parabola $y^2-4ax=0$ and are divided by it in the ratio $\lambda : 1$ have for their equation
$$4\left\{y'(y-y')-2a(x-x')\right\}^2\lambda+(y-y')^2(y'^2-4ax')(\lambda-1)^2=0.$$

8. Prove that, if the angle subtended by the chord of an ellipse at the focus be constant, the chord touches a conic with the same focus and directrix.

9. Two parabolas have the same vertex and their axes are perpendicular. Find the locus of a point from which a tangent to one is perpendicular to a tangent to the other.

10. $ABCD$ and $AEFG$ are two squares, $E$ being on $AD$ and $G$ on $BA$ produced through $A$. Prove that, if $FE$ produced meets $BC$ in $H$, then $AH$, $GC$, and $FD$ meet in a point.

## Problem Paper 9.

1. If
$$ax^2+2hxy+by^2+2gx+2fy+c=0$$
represents two straight lines, show that the product of the perpendiculars from $(x, y)$ on the straight lines is
$$\frac{f(x, y)}{\sqrt{(a-b)^2+4h^2}}.$$

2. Find the locus of a point such that the tangents drawn from it to two given circles shall have a constant ratio.

3. Find the length of the perpendicular from the point $(r_3, \theta_3)$ on the line joining the two points $(r_1, \theta_1)$, $(r_2, \theta_2)$.

4. Find the coordinates of the centre of the conic
$$x^2-xy+y^2-7x+8y+18=0;$$
transform the equation to parallel axes through the centre, and draw the curve.

5. Prove that, if from any point $P$ on the ellipse $x^2/a^2+y^2/b^2=1$ perpendiculars $PM$, $PN$ be let fall upon the axes, the equation of the locus of the foot of the perpendicular let fall upon $MN$ from the centre is
$$\frac{1}{r^2}=\frac{\sec^2\theta}{a^2}+\frac{\operatorname{cosec}^2\theta}{b^2}.$$

6. Prove that all chords of the parabola $y^2-4ax=0$ which subtend a constant angle $\alpha$ at the vertex touch the conic
$$(x-4a)^2+4y^2+4\cot^2\alpha\,(y^2-4ax)=0.$$

7. Show that the locus of points at which the ellipse $x^2/a^2+y^2/b^2=1$ subtends an angle 60° is
$$3(x^2+y^2-a^2-b^2)^2=4(x^2b^2+y^2a^2-a^2b^2).$$

8. If $PQ$ is a normal chord of the parabola $y^2=4ax$, and if $S$ is the focus, prove that the locus of the centroid of the triangle $SPQ$ is
$$36ay^2(3x-5a)-81y^4=128a^4.$$

9. Show that the greatest angle between two corresponding tangents to the ellipse and the auxiliary circle is
$$\sin^{-1}\frac{a-b}{a+b}.$$

10. Two points $P$ and $Q$ are taken one on each of the coordinate axes and equidistant from a fixed point $(a, b)$. Find the equation of the locus of the middle point of $PQ$; and trace the curve.

## Problem Paper 10.

1. A square being described upon the part of the straight line $x/a + y/b = 1$ intercepted between the coordinate axes, find the equations of the sides and the coordinates of the point of intersection of the diagonals.

2. If three circles, whose centres are fixed, have for their radii $r_1 + k$, $r_2 + k$, $r_3 + k$, find the locus of the radical centre when $k$ varies.

3. The two parabolas $y^2 = 4ax$, $x^2 = 4by$ intersect at the origin $O$ and at $P$. Show that the tangents of the inclinations to either axis of $OP$ and the tangents at $P$ are in geometrical progression, and that the two triangles bounded by the two tangents and one of the axes are equal in area.

4. Show that from any point $(x', y')$ three normals, real or imaginary, can be drawn to the parabola $y^2 - 4ax = 0$, and that, when the normals are real, the circle passing through their feet passes also through the vertex. Show that the equation of the circle is

$$x^2 + y^2 - \frac{y'y}{2} - (2a + x')\,x = 0.$$

5. Prove that, if $\alpha$, $\beta$ be the inclinations to the axis of $x$ of two conjugate diameters of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then
$$b \tan\alpha \tan\beta + h\,(\tan\alpha + \tan\beta) + a = 0.$$

6. $S$, $S'$ are the foci of an ellipse, $Q$, $Q'$ are points on it, on the same side of the major axis and such that $SQ$, $S'Q'$ are parallel, being inclined at an angle $\theta$ to $SS'$. Prove that $QQ'$ and the tangent at the point whose eccentric angle is $\theta$ meet on the major axis.

7. Show that the locus of the point of intersection of equal chords of a parabola drawn in given fixed directions is a straight line.

8. Find the locus of the centre of an equilateral hyperbola which passes through three given points.

9. Prove that through any point in the plane of an ellipse a confocal ellipse and a confocal hyperbola can be drawn, and that, if $a$, $a'$ be their respective semi-major axes, the conjugate semi-diameter in the ellipse of the radius to the point is $(a^2 - a'^2)^{\frac{1}{2}}$. Show that the area of the parallelogram formed by the tangents to the hyperbola at its four points of intersection with the confocal ellipse is

$$2\,\frac{a'b'}{ab}\,(a^2 - b^2).$$

10. Show that, if a circle touch an ellipse in two points, the tangent to the circle from any point on the ellipse bears a constant ratio to the perpendicular from the point on the common chord of the ellipse and the circle.

## Problem Paper 11.

1. Two variable points $P$, $Q$ are taken on the sides $CB$, $CA$ of a given triangle such that $CP : PA = BQ : QC$. Find the locus of the mid-point of $PQ$.

2. Find the length of the chord of intersection of the circles

$$y^2 = 4x - x^2, \quad x^2 = 8y - y^2,$$

and the tangents of the angles between the circles at either intersection.

3. Show that the equation to the locus of the middle points of a chord of a rectangular hyperbola is of the form

$$(x^2 + y^2)(a + xy) + bxy = 0,$$

if the chord is of constant length.

4. $P$ and $Q$ are two points on a parabola the tangents at which meet in $T$ and the normals in $N$. Prove that the projection of $TN$ on the axis is equal to the sum of the distances of $P$ and $Q$ from the directrix.

5. Find the length of the latus rectum of the parabola whose equation is $9x^2 - 24xy + 16y^2 + 44x + 8y + 5 = 0$.

6. Trace the curve $2x^2 - 3xy - 2y^2 + 5y + 2 = 0$, and draw its asymptotes.

7. Prove that, if $(x, y)$ and $(x', y')$ are any two points on the ellipse $x^2/a^2 + y^2/b^2 = 1$, then

$$\frac{x}{a^2}\frac{1}{y + y'} + \frac{y}{b^2}\frac{1}{x + x'} = \frac{1}{xy' + x'y},$$

and hence show that the locus of the middle points of parallel chords is a straight line.

8. $PQ$ is a chord of an ellipse, normal at $P$. The points on the auxiliary circle corresponding to $P$, $Q$ are $p$, $q$. Prove that the angle $pCq$ must exceed

$$2\tan^{-1}\frac{2\sqrt{1-e^2}}{e^2},$$

where $e$ is the eccentricity of the ellipse.

9. If an ellipse be inscribed in a square, prove that the axes of the ellipse coincide in direction with the diagonals of the square.

10. A parabola touches two straight lines $OA$, $OB$ in $A$ and $B$. Show that the portions of any chord which has its middle point on $AB$ intercepted between $OA$, $OB$ and the parabola are equal.

## Problem Paper 12.

1. Show that one of the bisectors of $ax^2+2hxy+by^2=0$ passes through the intersection of the two straight lines

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

if
$$h(g^2-f^2)=gf(a-b).$$

2. Find the cosine of the angle at which the two circles

$$x^2+y^2+2gx+2fy+c=0 \quad \text{and} \quad x^2+y^2+2g'x+2f'y+c'=0$$

intersect.

3. Tangents are drawn to the parabola $y^2=4ax$ at points whose abscissæ are in the ratio $\mu:1$. Show that the locus of their intersection is the parabola $y^2=(\mu^{\frac{1}{4}}+\mu^{-\frac{1}{4}})^2.ax$.

4. Prove that the length of the chord of the ellipse $x^2/a^2+y^2/b^2-1=0$ which lies along the line $lx/a+my/b-1=0$ is

$$\frac{2}{l^2+m^2}\sqrt{l^2+m^2-1}\,.\sqrt{a^2l^2+b^2m^2}.$$

5. Prove that the lengths $p$ and $\varpi$ of the perpendiculars from the centre on the tangent and normal, respectively, at any point of an ellipse of semi-axes $a$ and $b$ are connected by the relation

$$p^2\varpi^2=(a^2-p^2)(p^2-b^2).$$

Show that the two normals on the same quadrant which are at a distance $\varpi$ from the centre make an angle $\cos^{-1}\varpi/(a-b)$ with one another.

6. If chords of the parabola $y^2=4ax$ touch a circle whose centre is at the focus, and radius $b$, the locus of their middle points will be given by the equation

$$\left\{y^2+2a(a-x)\right\}^2=b^2(y^2+4a^2).$$

7. Find the equation of the axis of the parabola

$$49x^2+126xy+81y^2+23x+11y+12=0.$$

8. Trace the curve $\qquad y^2+2xy+2y=1,$

and find the equation of the tangent from the origin.

9. Two lines intersect in $O$. On the first are taken three points, $A_1$, $A_2$, $A_3$; on the second three others, $B_1$, $B_2$, $B_3$. If the second line be now caused to rotate round $O$, the points $B_1$, $B_2$, $B_3$ moving with it, show that the area of the triangle formed by the joins $A_1B_1$, $A_2B_2$, $A_3B_3$ will vary as the sine of the angle at $O$.

10. Show that the tangent and ordinate at any point of the ellipse $b^2x^2+a^2y^2=a^2b^2$ are harmonic conjugates with respect to the chords which join the point to the extremities of the major axis.

## Problem Paper 13.

1. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines, show that the square of the distance of the origin from their point of intersection is $\dfrac{c(a+b)-f^2-g^2}{ab-h^2}$.

2. The axes $AB$, $AC$ are inclined at an angle of 60°. A circle touches $AB$ at $P$, and intercepts on $AC$ a chord whose length is equal to $AP$. Show that the locus of the centre is a straight line, and give the equation to the circle when $AP=c$.

3. Find the coordinates of the points of contact of the common tangents to the two hyperbolas

$$x^2-y^2=3a^2 \quad \text{and} \quad xy=2a^2.$$

4. The tangent at any point of a parabola makes an angle $\tan^{-1}\mu$ with the principal diameter, and $\delta$ is the intercept made by the curve on the normal at the point. Prove that

$$\text{the latus rectum} = \frac{\mu\delta}{(1+\mu^2)^{\frac{3}{2}}}.$$

5. Show that the locus of the middle points of normal chords of the ellipse $x^2/a^2+y^2/b^2=1$ is a curve whose equation is

$$(b^2x^2+a^2y^2)^2(b^6x^2+a^6y^2)=a^4b^4x^2y^2(a^2-b^2)^2.$$

6. In an ellipse $2abx^2+(a^2+b^2)y^2=(a+b)^2$,

find the eccentricity, two axes, area, and latus rectum, and find the equation to the tangent at the extremity of the latus rectum.

7. Draw the curve $8x^2+12xy+17y^2+12x+34y+17=0$.

8. If the perpendicular from a point on its polar to a conic passes through a fixed point on the axis of the conic, show that the locus of the point is a straight line.

9. Show that the envelope of the chords of the hyperbola $xy=a^2$ which subtend a given angle $\alpha$ at the point $(x', y')$ on the curve is the hyperbola $x^2x'^2+y^2y'^2=2a^2xy(1+2\cot^2\alpha)-4a^4\operatorname{cosec}^2\alpha$.

10. Prove that the equation $x+y=k+k\sqrt{xy}$ represents a conic section touching both axes. For what value of $k$ will this conic be a circle, and what will be the radius of the circle, the axes being rectangular?

## Problem Paper 14.

1. Through the origin $O$ the lines $OA$ and $OB$ are drawn, of lengths $a$ and $b$, and inclined to $Ox$ at 30° and 60° respectively: find the equation to $AB$.

2. Prove that, if $u \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ represent a pair of straight lines, the third pair of straight lines through the four points in which these lines meet the coordinate axes are

$$cu+4(fg-ch)\,xy=0.$$

3. If $y'$ be the ordinate of any point $P$ on a parabola of latus rectum $4a$, the equation of the circle described upon the focal distance $FP$ as diameter is $$x^2+y^2-\left(a+\frac{y'^2}{4a}\right)x-y'y+\frac{y'^2}{4}=0.$$

Show that this circle always touches a fixed right line.

4. If $TP$, $TQ$ be the tangents from the point $T(f, g)$ to the ellipse $x^2/a^2+y^2/b^2-1=0$, whose centre is $C$, prove that the area of the quadrilateral $TPCQ$ is $\sqrt{b^2f^2+a^2g^2-a^2b^2}$.

5. Prove that the equation of the tangents to a conic from a focus satisfies the condition for a circle.

6. Prove that the normals at the points where the line

$$x/a\cos\alpha+y/b\sin\alpha=1$$

intersects the conic $x^2/a^2+y^2/b^2=1$ meet at the point whose coordinates are $$-c^2\cos^3\alpha/a,\quad c^2\sin^3\alpha/b.$$

7. Trace the curve $15x^2-23xy-28y^2+x+29y-6=0$.

8. Discuss the curve

$$r+\frac{1}{r}=2\cos\theta+2\sin\theta,$$

and find the coordinates of the focus of the curve

$$(y-x)^2=y+x+1.$$

9. Show that the sum of the eccentric angles at the points where normals to $x^2/a^2-y^2/b^2=1$ from a given point meet the hyperbola is an odd multiple of two right angles.

10. A tangent is drawn to a rectangular hyperbola whose asymptotes are the coordinate axes. Its poles are taken with respect to a series of conics confocal with $x^2/a^2+y^2/b^2=1$. Prove that the polars of all these poles with respect to the conic $x^2/a'^2+y^2/b'^2=1$ meet in a point; also that the locus of such points is a rectangular hyperbola having the axes of coordinates as asymptotes, and conjugate to the given hyperbola if $4a'^2b'^2=(a^2-b^2)^2$.

## Problem Paper 15.

1. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two parallel straight lines, prove that $h^2 = ab$ and $bg^2 = af^2$, and the distance between the lines is
$$2\left\{\frac{g^2 - ac}{a(a+b)}\right\}^{\frac{1}{2}}.$$

2. Determine the position of the radius vector drawn through the focus of an ellipse cutting the curve most obliquely.

3. Tangents are drawn to a parabola, whose equation is $y^2 - 4ax = 0$, from the point $(4a, 6a)$, and the points of contact are joined. Find the area of the triangle thus formed

4. If the tangents to a parabola include a constant angle of 135°, find the locus of the point of intersection of the diagonals of the quadrilateral formed by the tangents and the focal distances of their points of contact.

5. A circle of constant radius passes through the vertex of a parabola. Show that the normals to the parabola at the other three points of intersection of the circle and parabola meet in a point, and that the locus of this point, for different positions of the circle, is an ellipse of eccentricity $\frac{1}{2}\sqrt{3}$.

6. Show that the locus of the middle points of normal chords to the parabola $y^2 = 4ax$ is
$$2axy^2 = y^4 + 4a^2y^2 + 8a^4.$$

7. Trace the curves

$$\text{(i.)}\ 144x^2 + 120xy + 25y^2 + 7x + 17y = 0;$$
$$\text{(ii.)}\ (3x - 4y + a)(4x + 3y + a) = a^2.$$

8. If $x_1$, $x_2$, $x_3$, $x_4$ be the abscissæ of four points on the hyperbola, the normals at which meet in a point, prove that
$$(x_1 + x_2 + x_3 + x_4)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right) = 4.$$

9. If $(x_0, y_0)$ is the middle point of a chord of an ellipse referred to the centre as origin, then $\dfrac{x_0}{\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}}$, $\dfrac{y_0}{\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}}$ are the coordinates of the pole of the chord, $a$ and $b$ being the semi-axes of the ellipse.

10. The locus of a point $P$ such that the tangent $PQ$ to the conic $ax^2 + by^2 = 1$ subtends a right angle at the centre is
$$a/x^2 + b/y^2 = (a-b)^2.$$

If $h$, $k$ be the coordinates of $Q$, the point of contact, then the chord of contact of $P$ with regard to the conic is
$$ax/h - by/k = a - b.$$

# ANSWERS TO PART I.

(Pages 1–54.)

**3.** 13 miles, 10 miles. **4.** 25. **5.** $\sqrt{(13)}$.

**6.** $x^2+y^2=a^2$. **7.** $(x-d)^2+(y-e)^2=a^2$.

**9.** $(-\frac{77}{12}, -\frac{5}{4})$. **10.** $(9\frac{1}{4}, 3\frac{1}{4})$. **11.** 4 : 7.

**12.** $\{(n-r)x_1+rx_2\}/n$, $\{(n-r)y_1+ry_2\}/n$.

**14.** (*a*) 7, (*b*) $\frac{1}{2}(a^2+b^2)$, (*c*) $\frac{1}{2}p(m-n)-\frac{1}{2}q(l-m)$. **15.** 2.

**18.** (*a*) $-1$, $-1$; (*b*) $-1$, $-1$; (*c*) $\frac{3}{2}$, $-\frac{1}{2}\sqrt{3}$.

**19.** (*a*) 2, $\frac{5}{4}\pi$; (*b*) $\sqrt{3}$, $\frac{5}{3}\pi$; (*c*) 5, $\pi$.

**20.** (*a*) $\sqrt{7}$, (*b*) 2. **21.** (*a*) $\frac{1}{2}$, (*b*) $\frac{5}{4}\sqrt{3}$.

**27.** $x^2+y^2=2$. **28.** $x^2+y^2\mp 2y=0$.

**29.** $y=x+a$, where $a$ is the given distance.

**30.** (*a*) Straight line through origin and point (1, 2); (*b*) circle with origin as centre and radius 4; (*c*) two straight lines joining origin with points (1, 2), (1, $-2$); (*d*) two straight lines parallel to $OX$ and at equal distances 2 above and below it respectively; (*e*) a curve similar to that in Part II., § 44, Ex. ii.; (*f*) the two axes; (*g*) straight line cutting axes in points (2, 0), (0, 3); (*h*) axis of $y$ and a straight line through origin and point (1, 1).

**33.** $y=\pm x\sqrt{3}+2$; $y=\pm x\sqrt{3}-2$; $(-\frac{2}{3}\sqrt{3}, 0)$, $(\frac{2}{3}\sqrt{3}, 0)$.

**34.** 30°, 60°. **37.** $x+y=\pm 2$. **38.** $x-y=1$.

**39.** $y=4$, $2x+3y=0$; $2x-y=0$. **40.** $2x+y=6$ or $x+2y=6$.

**41.** $x-y+1=0$; $3\sqrt{2}$. **42.** 28 : 11. **44.** 45°.

**45.** Same side. **47.** (*a*) $\frac{3}{5}x+\frac{4}{5}y=2$, (*b*) $-\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}}{2}=2\sqrt{3}$.

**50.** $(a-b)^2/\sqrt{(a^2+b^2)}$.

**52.** $(-4, -1)$, $(3, -2)$, $(2, 1)$; $2\sqrt{(10)}$, $\sqrt{(10)}$, $2\sqrt{2}$.

**53.** 30°. **54.** 30°. **55.** 45°.

**56.** $x+7y+22=0$; $y-7x-4=0$. **57.** $x+11y=0$.

**58.** $x=0$, $\sqrt{3}y-x=0$; $(0, -3)$, $(-\frac{3}{2}\sqrt{3}, -\frac{3}{2})$. **59.** $(10\frac{2}{3}, 0)$.

**60.** $x=(h+mk-mb)/(1+m^2)$, $y=(mh+m^2k+b)/(1+m^2)$.

**61.** $k=-4\frac{1}{2}$. **62.** $\frac{9}{4}$. **63.** 30°. **64.** 90°.

**65.** 1, 2. **66.** $y=1$. **68.** $21x+13y=76$.

**69.** $7x-21y+50=0$, $22x-11y+27=0$, $15x+10y-23=0$; $(-\frac{17}{385}, \frac{211}{385})$.

**70.** $10x-15y+41=0$, $21x+7y=20$, $11x+22y=49$.

**71.** $4x-3y+2=0$, $7x+y=9$.

**74.** $21x+77y=136$, $99x-27y+46=0$.

**75.** (i.) $y-3x+6=\pm\sqrt{2}(y-2x+4)$, $y=0$.
(ii.) $6x+y=5$, $x-3y+10=0$, $y+13x=0$.

**76.** $9x-7y=3$, $16x+28y+51=0$, $7x+7y+15=0$.

**79.** $x=2y$, $x=3y$.

**80.** (*a*) The two axes; (*b*) two straight lines both coinciding with the axis of $y$; (*c*) two "imaginary" straight lines, $x+y\sqrt{(-1)}=0$ and $x-y\sqrt{(-1)}=0$, passing through the origin, which is the only real point on the locus; (*d*) axis of $x$ and line $x+y=0$; (*e*) two coincident straight lines $x-y=0$; (*f*) two imaginary straight lines, $y=\pm\sqrt{(-1)}$, parallel to axis of $x$; (*g*) two imaginary straight lines, $x-a=\pm(y-b)\sqrt{(-1)}$ passing through the point $(a, b)$; (*h*) $x=2$ and $y=3$.

**81.** $\text{Tan}^{-1}\frac{1}{3}$. **82.** $y=3x$, $y+2x=0$; $45°$. **83.** $60°$.

**87.** (*a*) $x^2+xy=0$; (*b*) $xy-y^2=0$; (*c*) $xy=0$; (*d*) $x^2-y^2=0$.

**88.** $bx+ay=0$. **89.** $x\cos\alpha+y\sin\alpha=0$.

**90.** $x^2+y^2=f^2+g^2-c$. **93.** $x-y+1=0$, $x+y=2$.

**94.** $3x+y=1$, $x-y+1=0$; $(0, 1)$.

**95.** $(2, 1)$; $x-y=1$; $x-4y+2=0$.

**97.** $(\frac{23}{7}, \frac{3}{7})$, $x^2+xy+2y^2=\frac{4}{7}$.

**98.** (i.) $x-1=0$; (ii.) $y+1=0$; (iii.) $lx+my-l+m+1=0$; (iv.) $x^2-y^2-2x-2y=0$; (v.) $3x^2-4xy+y^2-10x+6y+8=0$, (vi.) $2x^2+3xy+4y^2-2x+6y+6=0$.

**99.** (i.) $x=\frac{1}{2}(x'\sqrt{3}-y')$, $y=\frac{1}{2}(x'+y'\sqrt{3})$;
(ii.) $x=-(1/\sqrt{2})(x'-y')$, $y=-(1/\sqrt{2})(x'+y')$.

**100.** $x=(1/\sqrt{5})(2x'-y')$, $y=(1/\sqrt{5})(x'+2y')$;
$x'=(1/\sqrt{5})(2x+y)$, $y'=(1/\sqrt{5})(-x+2y)$.

**101.** $3x^2=y^2$. **102.** $x=-y'$, $y=x'$. **103.** $y^2=4x$.

**104.** $x^2-8xy-4y^2=0$. **105.** $39x^2-96xy+11y^2=0$.

**106.** $3x=2y$; $y=3$ meets the curve (a circle) in two coincident points. **107.** $(-\frac{5}{3}, 3)$, $(3, -\frac{1}{2})$.

**109.** $(5, \pi)$. **110.** $x^2+y^2=4$. **111.** The two axes.

**112.** Straight line parallel to axis of $x$ at distance 1 below it.

**113.** Straight line through origin and (1, 2).

**114.** $x/(x_1+y_1\cos\omega)+y/(x_1\cos\omega+y_1)=1$.

**115.** $p/(x+y\cos\omega)+q/(x\cos\omega+y)=1$.

**118.** (i.) $\infty$ ; (ii.) $\frac{1}{2}\sqrt{41}$ ; (iii.) $\sqrt{3}$ ; (iv.) $\frac{1}{2}\sqrt{-3}$. The lines are imaginary. **119.** $16x^2+37xy+y^2=0$.

**121.** $(-\frac{3}{2},\frac{1}{2})$, $(-\frac{1}{2},-\frac{1}{2})$, $(-6,5)$, $(-\frac{7}{6},\frac{5}{6})$, $(0,2)$, $(-\frac{7}{3},\frac{19}{6})$; $32x+10y+43=0$, $5x-y+2=0$, $25x+29y+5=0$.

**123.** $\frac{85}{36}$ or $-\frac{5}{4}$. **124.** $2x-3y+2=0$, $3x+4y=7$.

**125.** Through $(1/c, 1/c)$, where $c$ is the constant.

**126.** $(-7, 3)$. **129.** $x=\frac{1}{5}(4x'-3y')$, $y=\frac{1}{5}(3x'+4y')$.

**131.** $3x-4y=-y'/\sqrt{2}$, $x-y+1=-x'/5$;
$y=y'/\sqrt{2}-3x'/5-3$, $x=y'/\sqrt{2}-4x'/5-4$.

**133.** If lengths of $AB$, $BC$ are $p$ and $q$, and $\alpha$ be $\angle BAD$, $AC$ is

$$\frac{q\sin\alpha}{q\cos\alpha+p},\ \theta=\tan^{-1}\{q\sin\alpha/(q\cos\alpha+p)\},$$

and $BD$ is $r\{p\sin\theta-q\sin(\theta-\alpha)\}=pq\sin\alpha$.

**134.** $\frac{1}{224}$. **135.** $r^2\sqrt{(h^2-ab)}/(aq^2-2hpq+bp^2)$.

**137.** $(ap^2+2hpq+bq^2)/\sqrt{(4h^2+\overline{a-b}^2)}$.

**139.** A straight line throught the point of intersection of the fixed straight lines.

**140.** If $a$ = base and $\alpha$ = difference of base angles, then, taking base as axis of $x$, locus of vertex is $x^2+y^2-ax-ay\cot\alpha=0$.

### Examination Paper (Page 55).

**1.** $x^2+y^2-ay\cot A=\frac{1}{4}a^2$, taking base as axis of $x$, and its mid-point as origin.

**2.** (i.) Straight line inclined at 150° to initial line, and distant $\frac{1}{2}a\sqrt{3}$ from pole ; (ii.) Straight line parallel to initial line and distant $a$ from it.

**3.** $-C/C'$. **4.** $10x+7y=0$, $13x+8y-11=0$.

**5.** $\frac{8}{15}$. **7.** $x^2-2\sqrt{3}xy-y^2=0$, $xy=0$.

**8.** $a\sin^2\omega\,.\,x^2+2\sin\omega\,(h-a\cos\omega)xy+(a\cos^2\omega-2h\cos\omega+b)y^2=0$

---

# ANSWERS TO PART II.

### Chapter I. (Pages 1–14.)

**2.** (a) $x^2+y^2-x-y=0$. (b) $x^2+y^2-6x+4y+4=0$.
(c) $x^2+y^2+2y=0$.

**3.** (a) $(-2, 2)$, 3 ; (b) $(8, 8)$, 10 ; (c) $(-a, a)$, $a$ ;
(d) Mid-point of line joining $(a, b)$, $(c, d)$. Rad. = $\frac{1}{2}$ distance between the points ; (e) $(0, a)$, $a$.

**4.** $60^\circ$; $(1, 2)$; $3$.

**5.** $(-g+f\sin\omega)/\sin\omega$, $(-f+g\sin\omega)/\sin\omega$;
$\sqrt{(g^2-2fg\cos\omega+f^2-c)}/\sin\omega$.

**6.** (a) $7x^2+7y^2-39x-15y+32=0$; $x^2+y^2=\frac{425}{98}$.
(b) $x^2+y^2-hx-ky=0$; $x^2+y^2=\frac{1}{4}(h^2+k^2)$.
(c) $x^2+y^2-12(x+y)+47=0$; $x^2+y^2=25$.
(d) $x^2+y^2-(a+b)(x+y)+2ab=0$; $x^2+y^2=\frac{1}{2}(a-b)^2$.

**7.** (a) $x^2+y^2-ax-by=0$;
(b) $x^2+y^2-2hx-2ky=p^2+q^2-2hp-2kq$.

**8.** $x^2+y^2-ay=0$; $(0, \frac{1}{2}a)$, $\frac{a}{2}$. **9.** $(x-a)^2+y^2=0$; $(a, 0)$.

**11.** $x^2+y^2-4x-2y-20=0$; $(2, 1)$, $5$.
**12.** (a) $(3, \frac{1}{5}\pi)$, $5$; (b) $(2, \frac{1}{6}\pi)$, $3$. **14.** $y=4$.
**15.** $2x+3y=13$, $2x-3y=13$. **16.** $3y-4x=24$, $3x-4y=24$.
**23.** $12x-5y=0$. **24.** $y=2$.
**25.** $(-a, 0)$, $(0, -a)$. **26.** $x^2+y^2=10$. **27.** $x^2+y^2=4$.
**28.** (a) $(-1, -1)$; (b) $-\frac{3}{2}\sqrt{3}$, $\frac{9}{2}$. **30.** $y=x+1\pm\sqrt{2}$.
**31.** $3x+4y\pm 25=0$. **32.** (a) $y+\sqrt{3}x=\pm 2\sqrt{2}$.
(b) $x+\sqrt{3}y=\pm 2\sqrt{2}$; (c) $5x-12y=\pm 13\sqrt{2}$;
**33.** $a^2b^2(x^2+y^2)-c^2(bx+ay)^2=0$. **35.** $(\frac{60}{13}, \frac{25}{13})$, $(-\frac{60}{13}, \frac{25}{13})$.
**36.** $x^2+y^2=b^2-a^2$. **37.** $(\frac{9}{5}, \pm\frac{12}{5})$, $3x\pm 4y=15$, $4x\mp 3y=0$.
**39.** $3x+4y+3a+4b\pm 5\sqrt{(a^2+b^2-c)}$. **40.** $\sqrt{3}y=x\pm 10$.
**42.** $(2\sqrt{3}, 30^\circ)$, $(2\sqrt{3}, -30^\circ)$. **45.** $x^2+2xy\cos\alpha+y^2-px-qy=0$.
**46.** $b^2k^2+4ak=4$. **47.** $\sqrt{(a^2+b^2-2ab\cos\overline{\alpha-\beta})}/\sin(\alpha-\beta)$.

## Chapter II. (Pages 15–33.)

**1.** (a) $3x-5y=2$; (b) $y-3x=14$; (c) $y=14$.
**2.** (a) $(21, -7)$; (b) $(1, -1)$; (c) $(6, 0)$.
**3.** $(la^2, ma^2)$. **4.** $6x+5y=48$. **6.** (a) $5$; (b) $1$; (c) $\frac{1}{2}\sqrt{2}$.
**8.** $4x-3y=0$. **9.** $10x-7y=0$. **10.** $x-3y=2$.
**11.** $32x+8y=9$. **12.** $4x+y=1$. **13.** $(1, 2)$.
**14.** $(2, 0)$, $(8, 0)$. **16.** $(x-5)(y-5)=0$.
**22.** $a^2(x^2+y^2)=(xx_1+yy_1)^2$.
**24.** $x+y=3$, $3x-2y+1=0$, $x=1$; $(1, 2)$.
**26.** $x^2+y^2-hx-ky=0$. **28.** $(1, 1)$.
**30.** Circle on radius through point on diameter.
**31.** $x^2+y^2+\frac{5}{2}x+2y=0$. **32.** $r^2(x^2+y^2)=(qx-py)^2$.

**34.** $c$ positive, $p$ and $q$ of same sign. **39.** $\sqrt{\{4r^2-2(c-d)^2\}}$.
**40.** $(x-r)^2+(y-r)^2=r^2$. **41.** $x^2+y^2-bx=(bx-ay-ab)^2/8ab$
**42.** $x^2+y^2+2\dfrac{Gc-Cg}{c-C}x+2\dfrac{Fc-Cf}{c-C}y=0$.

**44.** Taking radical axis as axis of $x$ and its mid-point as origin, $2c$ as length of radical axis between points of intersection of circles, and $m$ as tangent of given direction with axis of $x$, locus is $x^2-y^2-c^2=\pm 2mxy$.

## Examination Paper A. (Page 34.)

**1.** $x^2+y^2=a^2$. **2.** $(a, 0)$. **5.** $\frac{60}{13}, \frac{25}{13}$.
**6.** $a(1+\cos\alpha)$ or $-a(1-\cos\alpha)$.
**8.** The circle $r=\rho\cos(\theta-\alpha)$ where $(\rho, \alpha)$ are the polar coordinates of the centre of the given circle.

## Chapter III. (Pages 35–48.)

**2.** (i.) $2a$; (ii.) $7$; (iii.) $\frac{1}{7}$. **3.** $y^2=12x$. **4.** (i.) $10, 10$; (ii.) $30, 10\sqrt{3}$.
**10.** $\frac{1}{3}$. **11.** 8. **12** $(-\frac{3}{2}, 0)$; focus, $(-1, 0)$; directrix, $x+2=0$.
**13.** $(0, -2)$, $x=0$. **15.** Outside former, within latter.
**17.** (i.) $(1, 4)$, $(\frac{3}{2}, 4)$, $x=\frac{1}{2}$; (ii.) $(-2, -3)$, $(-\frac{5}{2}, -3)$, $2x+5=0$.
**18.** $y^2=4ax$. **19.** $x^2-2xy+y^2-4y+6=0$. **20.** $4a$, $\sqrt{2}$.
**21.** $-\frac{1}{4}, \frac{1}{2}$. **22.** $-\frac{5}{2}, 2$; axis, $y=2$; tangent at vertex, $2x+5=0$.
**23.** Equation may be written $(x-a/2)^2=b(y+a^2/4b)$.
**25.** $(1, -1)$, $(1, -\frac{3}{4})$, $4y+5=0$. **27.** $x^2-2xy+y^2-8y=0$.
**28.** $2\sqrt{2}$. **29.** $\frac{1}{2}(11\mp\sqrt{85})$, $\frac{1}{2}(-5\pm\sqrt{85})$. **30.** $c=mb+a/m$.
**32.** The abscissæ are $\frac{1}{9}(-5\pm\sqrt{-2})$, so the points are imaginary.
**33.** $\frac{11}{2}, -\frac{5}{2}$. **34.** $8\sqrt{3a}$. **35.** $4a(2\pm\sqrt{3})$.
**40.** $x^2+2xy+y^2-6x-2y+4=0$. **41.** $x^2+2xy+y^2-2x-6y+4=0$.

## Examination Paper I. (Page 49.)

**1.** § 17; $g^2=ac(bc+2f)$.
**2.** $x/c=y/b$ or $x/c+y/b=1$, where $AB=c$ and $AC=b$.
**3.** $\frac{1}{2}\tan^{-1}\{2h/(a-b)\}$; $a'+b'=a+b$. **4.** § 36.
**6.** (i.) $4y-3a=0$; (ii.) $(\frac{5}{2}a, -\frac{3}{4}a)$, $(-\frac{1}{2}a, -\frac{3}{4}a)$; $-\frac{1}{3}a$.
**7.** $x^2-2xy+y^2-4ay-2a^2=0$; $a\sqrt{2}$. **8.** $(an^2, -2an)$. **9.** § 52.

## Chapter IV. (Pages 50–68.)

**1.** $7x^2-2xy+7y^2-16y+8=0$. **2.** $x^2/a^2+y^2/b^2=1$.
**4.** (i.) $\pm\sqrt{3}$; (ii.) $\pm\frac{1}{4}\sqrt{3}$; (iii.) $\pm\frac{1}{2}$. **6.** On (i.), within (ii.).

**13.** (i.) $\frac{1}{2}\sqrt{3}$; (ii.) (0, 1), (4, 1); (iii.) (2, 0), (2, 2).
**14.** (i.) $\frac{2}{3}\sqrt{2}$; (ii.) (7, 2), (5, 2); (iii.) (1, 0), (1, 4).
**15.** (i.) $\frac{1}{2}\sqrt{3}$; (ii.) (2, 1), (−2, 1); (iii.) (0, 0), (0, 2).
**16.** (i.) $\frac{1}{2}\sqrt{3}$; (ii.) (0, −1), (0, 3); (iii.) (1, 1), (−1, 1).
**17.** (i.) $\frac{1}{2}\sqrt{2}$; (ii.) (1, 1), (−3, 1); (iii.) $(1, \sqrt{2}+1)$, $(1, 1-\sqrt{2})$.
**18.** (i.) $\frac{1}{15}\sqrt{165}$; (ii.) $(3, 2\sqrt{15}-4)$, $(3, -[2\sqrt{15}+4])$;
(iii.) (−1, −4), (7, −4).
**19.** (i.) $\frac{6}{13}\sqrt{26}$, $\frac{12}{31}\sqrt{31}$; (ii.) $\frac{1}{5}\sqrt{10}$, $\frac{2}{13}\sqrt{13}$; (iii.) $\frac{1}{5}\sqrt{10}$, $\frac{2}{7}\sqrt{7}$.
**20.** $\frac{1}{18}\sqrt{15}$, $\frac{1}{8}\sqrt{226}$. **22.** $\frac{12}{43}\sqrt{43}$, $\frac{12}{5}\sqrt{2}$, $\frac{8}{19}\sqrt{57}$.
**23.** $\frac{1}{4}\sqrt{7}$, $2l = \frac{9}{2}$. **24.** $7x^2-2xy+7y^2-18x-34y+39=0$, $2\sqrt{2}$.
**25.** $\frac{8}{3}\sqrt{2}$, $\frac{4}{3}\sqrt{6}$. **26.** $(\frac{1}{3}, \frac{4}{3})$, (3, 4). **27.** $(\frac{7}{3}, \frac{10}{3})$, $3(x+y)=29$.
**28.** $216x^2-24xy+209y^2-183x-94y-124=0$. **30.** $x^2/25+y^2/9=1$.
**31.** $\frac{4}{5}$, $2l = \frac{18}{5}$. **32.** $c = \pm\frac{1}{2}\sqrt{37}$; $(\mp\frac{6}{37}\sqrt{37}, \pm\frac{1}{74}\sqrt{37})$.
**33.** $y = x \pm \frac{1}{2}\sqrt{5}$. **34.** $\pm\dfrac{a^2\tan\alpha}{\sqrt{b^2+a^2\tan^2\alpha}}$, $\mp\dfrac{b^2}{\sqrt{b^2+a^2\tan^2\alpha}}$.
**35.** $(\frac{1}{2}\sqrt{3}\,x-\frac{1}{2}y+1)^2/9+(\frac{1}{2}x+\frac{1}{2}\sqrt{3}\,y+1)^2=1$. **38.** $(\frac{4}{5}, \frac{6}{5})$.
**41.** $\frac{1}{2}$, $\sqrt{3}$, $(1, \pm\frac{1}{3}\sqrt{3})$. **43.** The latter; both.
**47.** (i.) $\frac{1}{2}\sqrt{3}$, $(3\pm\frac{3}{2}\sqrt{3}, 0)$, $x = 3\pm 2\sqrt{3}$. (ii.) A circle; eccentricity = zero; foci at centre $(\frac{5}{3}, 0)$; directrices at infinity.
**48.** Take fixed lines as axes, $\theta$ inclination of bar to $OX$; then $x = a\cos\theta$, $y = b\sin\theta$, where $a$ and $b$ are lengths of the portions of bar. Eliminate $\theta$.
**49.** (0,0); $x\cos\alpha+y\sin\alpha-p=0$, $x\cos\alpha+y\sin\alpha+(1+e^2)/(1-e^2)p=0$.
**50.** $3x^2-2xy+3y^2-2(\sqrt{3}-1)ay=0$. **51.** $x^2/a^2-2x/a+y^2/b^2=0$.

## Chapter V. (Pages 69–87.)

**1.** $2xy-2x+2y-3=0$. **3.** $3x^2-y^2+16x+16=0$.
**4.** 1, $\sqrt{2}$; 2, $\sqrt{6}$; $\frac{1}{3}\sqrt{15}$, $\frac{1}{2}\sqrt{5}$; $\sqrt{(c/a)}$, $\sqrt{(c/b)}$.
**5.** $\sqrt{3}$; $\frac{1}{2}\sqrt{10}$; $\frac{1}{2}\sqrt{7}$; $\sqrt{a+b}/\sqrt{b}$.
**7.** (i.) $\sqrt{-6}$, $\frac{3}{2}\sqrt{-1}$; (ii.) $\frac{2}{11}\sqrt{11}$, $\frac{1}{3}\sqrt{6}$; (iii.) $\frac{2}{3}\sqrt{3}$, $\frac{1}{5}\sqrt{-10}$.
**8.** $\frac{2}{35}\sqrt{70}$, $\frac{1}{3}\sqrt{129}$. **13.** $\frac{1}{3}\sqrt{13}$; $(4, -\frac{1}{2})$, $(-2, -\frac{1}{2})$.
**14.** $\frac{1}{2}\sqrt{5}$; $(2\sqrt{2}-1, 3)$, $(-2\sqrt{2}-1, 3)$.
**15.** $\sqrt{5}$; $(\sqrt{7}-2, 1)$, $(-\sqrt{7}-2, 1)$. **16.** $\sqrt{10}$; $(\sqrt{3}, 3)$, $(\sqrt{3}, -3)$.
**17.** $2l = 24\frac{1}{2}$, $e = \frac{1}{4}\sqrt{65}$, $CS = \sqrt{65}$, $CX = \frac{16}{65}\sqrt{65}$. **18.** $2l = 8$.
**19.** $\frac{4}{3}$, $\frac{4}{3}\sqrt{3}$. **20.** $2\sqrt{2}$, $2\sqrt{-3}$.
**21.** $3x^2+10xy+3y^2-24x-16y+16=0$, $2l = 2\sqrt{10}$.
**25.** $4x^2/9-4y^2/55=1$. **26.** $\frac{8}{3}$, $2l = \frac{55}{3}$. **27.** $p = \frac{1}{2}\sqrt{5}$; $(-\frac{3}{2}\sqrt{5}, -\frac{3}{4})$.
**28.** $y = x\pm\frac{3}{20}\sqrt{390}$. **29.** Real. **30.** $\frac{1}{2}\sqrt{10}$, $\frac{1}{3}\sqrt{15}$.

**32.** 12 or $\frac{4}{5}$. **34.** $\frac{2}{3}\sqrt{3}$. **39.** $xy = \frac{1}{2}a^2$. **40.** $xy = \frac{25}{24}$.
**41.** $xy = c(a+b)/(4ab)$. **42.** $xy = 2\sqrt{2}$. **43.** $k = \pm 2c\sqrt{-m}$.
**44.** $m = \sin\theta/\sin(\omega-\theta)$ is negative, and therefore $\theta > \omega$.
**46.** $x^2+6xy+y^2-16x-8y+8 = 0$.
**47.** $2l = 2\sqrt{6}$. **51.** $(x+y-1)(x-y-2) = \frac{5}{2}$.
**55.** $x = 0,\ x+y = 0$; $x = 0,\ x-y = 0$; $y = 0,\ x+y = 0$.

## Examination Paper II. (Page 88.)

**1.** §§ 56, 62, 71, 74, 76, 83.
**2.** $e = \frac{1}{3}\sqrt{5}$; $A$ and $A'$ are $(1, \frac{1}{3})$, $(0, \frac{1}{3})$; $B$ and $B'$ are $(\frac{1}{2}, \frac{2}{3})$, $(\frac{1}{2}, 0)$; $2l = \frac{4}{9}$; equations of latera recta are $y = \frac{1}{2}+\frac{1}{6}\sqrt{5}$ and $y = \frac{1}{2}-\frac{1}{6}\sqrt{5}$. **3.** § 64. **4.** § 59.
**5.** $c$ is intercept on axis of $x$. **6.** § 69. **7.** $(-\frac{1}{2}, 2\pm 2\sqrt{5})$.
**8.** § 86. **9.** § 83. **10.** $(x+y+1)(2x-y+2) = \frac{81}{4}$.

## Chapter VI. (Pages 89–105.)

**1.** $(-8, 4)$. **2.** $(1, 1)$. **3.** $(\frac{1}{2}, 2)$.
**4.** Centre is on the line $2x+1 = 0$ at an infinite distance.
**5.** $(-\frac{3}{4}, \frac{5}{4})$; $3x^2+2xy-y^2+\frac{3}{4} = 0$. **6.** $(-\frac{1}{8}, -\frac{1}{2})$; $4x^2+y^2+\frac{11}{16} = 0$.
**7.** $(13, -4)$; $xy+2y^2+37 = 0$.
**9.** $-4x^2-\frac{8}{3}xy+\frac{4}{3}y^2 = 1$; $-\frac{64}{11}x^2-\frac{16}{11}y^2 = 1$; $-\frac{1}{37}xy-\frac{2}{37}y^2 = 1$.
**11.** $\frac{1}{2}\sqrt{2}$, $\frac{1}{2}$; $x+y = 0$, $x-y = 0$; $2x^2+4y^2 = 1$.
**12.** $\frac{1}{3}\sqrt{3}$, $\frac{1}{4}\sqrt{2}$; $2x+y = 0$, $x-2y = 0$; $3x^2+8y^2 = 1$.
**13.** $\sqrt{6}$, $\sqrt{2}$; $x+y = 0$, $x-y = 0$; $x^2/6+y^2/2 = 1$.
**14.** $\frac{1}{9}\sqrt{3}$, $\frac{1}{70}\sqrt{-70}$; $4x-9y = 0$, $9x+4y = 0$; $x^2/27-y^2/70 = 1$.
**15.** (i.), (ii.), (iii.), ellipses; (iv.), (v.), hyperbolas.
**18.** $x^2+2xy-2y^2+x+y+\frac{1}{4} = 0$.
**19.** $4x^2+4xy+7y^2+11x+9y+2\frac{47}{84} = 0$.
**20.** $3x^2+xy-y^2+2y-\frac{12}{13} = 0$. **21.** $x = 0$, $x+y = 0$.
**22.** $y = 0$, $x-y = 0$. **23.** $x+3 = 0$, $y+2 = 0$.
**24.** $x+1 = 0$, $x+y = 0$. **25.** $x-1 = 0$, $y+2 = 0$.
**26.** $x = 0$, $2x+y-1 = 0$. **27.** $x-y-2 = 0$, $3x+2y = 0$.
**28.** $(0, 0)$; $(0, 0)$; $(-3, -2)$; $(-1, 1)$; $(1, -2)$; $(0, 1)$; $(\frac{4}{5}, -\frac{6}{5})$.
**29.** $(\frac{14}{17}, \frac{39}{17})$. **30.** $25x^2-36xy+40y^2 = 52$. **31.** $x^2/3-y^2/2 = 1$.
**34.** Use result of § 104. **35.** $(3x-y+1)(x+y) = 6$.
**36.** $x(y-3)+1 = 0$. **37.** $(x+2y+7)(3x-y-4)+36 = 0$.
**38.** Axes bisect angles between the asymptotes; $x^2+2xy-y^2 = 0$.
**39.** $(x^2-y^2)/(A-B) = xy/H$. **40.** $2x^2+xy-y^2-3x+3y = 0$.

## Chapter VII. (Pages 106–111.)

**1.** Centre (1, 1). Equation of major axis referred to new centre is $x-y=0$. Lengths of semi-axes $\sqrt{5}$ and $\frac{1}{3}\sqrt{15}$ or 2·24 and 1·29. Intercepts on $OX = 1{\cdot}82$ and $-{\cdot}82$; on $OY = 1{\cdot}82$ and $-{\cdot}82$.

**2.** Centre $(\frac{13}{5}, \frac{9}{5})$. Major axis $x = 2y$. Lengths of semi-axes $\frac{1}{5}\sqrt{40}$, $\frac{1}{5}\sqrt{20}$ or 1·27 and ·89. Intercepts on $OX$ and $OY$ imaginary. Intercepts on $x = 2$ are 2·48 or ·85.

**3.** Centre $(-1, 3)$. Major axis $x = 0$. Lengths of semi-axes $\sqrt{5}$ and $\sqrt{2}$ or 2·24 and 1·41. Intercepts on $OX$ imaginary; on $OY = 4{\cdot}58$ and 1·42.

**4.** Centre $(\frac{4}{5}, \frac{6}{5})$. Major axis $2x+3y=0$. Lengths of semi-axes 1·47 and 1·03. Intercepts on $OX$ imaginary; on $OY = 2{\cdot}22$ and ·49.

**5.** Centre (1, 1). Major axis $x+3y=0$. Lengths of semi-axes 2·49 and 1·76. Intercepts on $OX = 3{\cdot}27$, $-{\cdot}73$; on $OY = 2{\cdot}80$, $-{\cdot}49$.

**6.** Centre $(\sqrt{2}, 1)$. Major axis $\sqrt{2}\,x+y=0$. Lengths of semi-axes 2 and 1. Intercepts on $OX = 2{\cdot}83$, ·94. Curve touches $OY$ at $y = 2$.

**7.** Centre (0, 0). Major axis $x+y=0$. Lengths of semi-axes 3·46 and 2. Intercepts on $OX = \pm 2{\cdot}45$, on $OY = \pm 2{\cdot}45$.

**8.** (*a*) $x-y=0$, $x+y-2=0$; (*b*) $x+1=0$, $y-3=0$;
(*c*) $x+3y-4=0$, $3x-y-2=0$.

**9.** Centre (0, 0). Major axis $x+y=0$. Lengths of semi-axes 1·41 and ·82. Intercepts on $OX = \pm 1$, on $OY = \pm 1$.

## Chapter VIII. (Pages 112–116.)

In the following (*a*) are coordinates of centre, (*b*) equations of transverse and conjugate axes referred to the centre as origin, (*c*) lengths of the semi-axes, (*d*) equations of asymptotes referred to the centre as origin, (*e*) intercepts on $OX$, (*f*) intercepts on $OY$;—

**1.** (*a*) $-1$, $-2$. (*b*) $3x+2y=0$, $2x-3y=0$. (*c*) 1, 2.
(*d*) $x-8y=0$, $7x-4y=0$. (*e*) 15·49, $-{\cdot}34$. (*f*) ·45, $-2{\cdot}57$.

**2.** (*a*) $\frac{1}{6}$, $\frac{1}{6}$. (*b*) $x-y=0$, $x+y=0$. (*c*) ·94. 1·63.
(*d*) $3{\cdot}73x+y=0$, ${\cdot}27x+y=0$. (*e*) 2·16, $-1{\cdot}16$.
(*f*) 2·16, $-1{\cdot}16$.

**3.** (*a*) 1, 1. (*b*) $3x-2y=0$, $2x+3y=0$. (*c*) ·58, ·71.
(*d*) $3{\cdot}25x+y=0$. ${\cdot}09x-y=0$. (*e*) 1·1, $-9{\cdot}1$. (*f*) 4·45, ·71.

**4.** (*a*) 2, 1. (*b*) $x+2y=0,\ 2x-y=0$. (*c*) 1, 1.
(*d*) $3x+y=0,\ x-3y=0$. (*e*) 2·78, $-1$·44. (*f*) 6·74, ·66.

**5.** (*a*) $-1,\ -1$. (*b*) $y=0,\ x=0$. (*c*) $\sqrt{2},\ \sqrt{3}$, or 1·41, 1·73.
(*d*) $y=\pm 1\cdot22x$. (*e*) ·63, $-2$·63. (*f*) Imaginary.

**6.** (*a*) $-\frac{2}{3},\ -\frac{1}{2}$. (*b*) $y=0,\ x=0$. (*c*) $\sqrt{3},\ \sqrt{3}$ or 1·73, 1·73.
(*d*) $x-y=0,\ x+y=0$. (*e*) 1·13, $-2$·45. (*f*) Imaginary.

**7.** (*a*) 0, 0, (*b*) $x-3y=0,\ 3x+y=0$. (*c*) ·71, 1·41.
(*d*) $7x-y=0,\ x+y=0$. (*c*) $\pm$·76. (*f*) Imaginary.

**8.** (*a*) 0, 0. (*b*) $3x-y=0,\ x+3y=0$. (*c*) ·58, 1·73.
(*d*) $y=0,\ 3x+4y=0$. (*e*) Infinite. (*f*) $\pm$·61.

**9.** (*a*) $-\sqrt{3}-\frac{1}{2},\ \frac{\sqrt{3}}{2}-1$, *i.e.* $-2$·23, $-$·14.
(*b*) $x=\sqrt{3}\,y,\ \sqrt{3}\,x+y=0$, (*c*) $\sqrt{2}+\frac{1}{2}\sqrt{6}$, *i.e.*, 2·64.
(*d*) $y=(3\pm\sqrt{2})\,x$, *i.e.* $y=3\cdot73x$ and $y=-\cdot27x$.
(*e*) $-2\sqrt{3}+1\pm\sqrt{14}$, *i.e.* 1·28 or $-6$·2.
(*f*) $\sqrt{3}+2\pm\sqrt{6}$, *i.e.* 6·18 or 1·28.

## Chapter IX. (Pages 117–123.)

**1.** $y=0,\ 3x+2=0$. **2.** $2y-1=0,\ 4x+5=0$.
**3.** $2x-1=0,\ 4y+5=0$. **4.** $x+y=0,\ x-y+1=0$.
**5.** $x+y+1=0,\ x-y-1=0$. **6.** $3x+4y-5=0,\ 4x-3y+7=0$.
**7.** 3, 1, 1, $\frac{3}{2}\sqrt{2},\ \frac{3}{2}\sqrt{2},\ \frac{17}{5}$. **8.** Two parallel straight lines $x+y=\pm 2$.
**9.** $x-y+1=0,\ x-y+6=0$.
**10.** Two straight lines parallel to $px+qy=0$.
**11.** Two coincident straight lines.

## Chapter X. (Pages 124–126.)

**2.** $2x-y+3=0,\ x+2y=0;\ 2\sqrt{5}$.
**3.** $x+3y=0,\ 3x-5y+25=0;\ \frac{1}{2}\sqrt{10}$.
**4.** $2x+3y+1=0,\ 3x-2y-10=0;\ \frac{2}{13}\sqrt{13}$.
**5.** $5x-4y+4=0,\ 4x+5y=0;\ \frac{3}{41}\sqrt{41}$.
**6.** $7x+9y+1=0,\ 9x-7y+11=0;\ \frac{1}{130}\sqrt{130}$.

## Chapter XI. (Pages 127–138.)

**1.** A pair of straight lines $2x+4y-7=0,\ 2x-3y+2=0$ passing through the point $(\frac{13}{14},\ \frac{20}{14})$.

**2.** An ellipse with origin as centre. Semi-axes = 4·9 and 4. Equations of axes $x-2y=0$, $2x\pm y=0$. Intercepts on $OX=\pm 4{\cdot}67$, on $OY=\pm 4{\cdot}13$.

**3.** A parabola, axis $3x-4y+2=0$, tangent at vertex $4x+3y+\frac{1}{8}=0$, latus rectum = 1·6. Curve is on side of tangent opposite to origin. Intercepts on $OX=-4{\cdot}7$, $-{\cdot}1$; on $OY$ imaginary. Curve passes through (2, ·7), (2, −4·2).

**4.** An ellipse with $2x+y-1=0$ as major axis and $x-2y+3=0$ as minor axis. Lengths of semi-axes = 2, $1\frac{1}{3}$. Intercepts on $OX=1{\cdot}1$, $-{\cdot}8$; on $OY=3{\cdot}09$, $-{\cdot}45$.

**5.** A rectangular hyperbola with $(\frac{14}{17}, \frac{39}{17})$ as centre, with principal axis parallel to $3{\cdot}6x+y=0$. Lengths of semi-axes = ·89. Asymptotes parallel to $y=1{\cdot}77x$, $y=-{\cdot}57x$. Intercepts on $OX=4{\cdot}7$, ·3; on $OY=3{\cdot}16$, ·44.

**6.** A hyperbola with $x-2y+1=0$, $x+2y-3=0$ as asymptotes. Centre is (1, 1). One branch of the curve lies on same side of both asymptotes as the origin. Semi-transverse axis = $\frac{1}{2}\sqrt{5}$, semi-conjugate axis = $\sqrt{5}$. Intercept on $OX$ imaginary, on $OY=2{\cdot}22$, $-{\cdot}22$. Curve also passes through (2, 2·22), (2, −·22), (3, 2·5), (3, −·5), (−1, 2·5). (−1, −·5).

**7.** A hyperbola with centre as origin and $3x+5y=0$ as transverse axis. Semi-transverse axis $=\frac{1}{3}\sqrt{3}={\cdot}58$, semi-conjugate axis $=\frac{1}{2}\sqrt{2}={\cdot}71$. Asymptotes are $y=-6{\cdot}88x$, $y={\cdot}36x$. Intercepts on $OX=\pm{\cdot}77$, on $OY$ imaginary. The curve passes through (1, ·15), (1, −6·67), (2, ·65), (2, −13·7), (·33, −1), (−2·96, −1), (−·33, 1), (2·96, 1).

**8.** A rectangular hyperbola with (5, 2) as centre and $y=2$, $x=5$ as axes. Semi-axes $=\sqrt{28}=5{\cdot}29$. Asymptotes are $x-y=3$, $x+y=7$. Intercepts on $OX=10{\cdot}66$, $-{\cdot}66$; on $OY$ imaginary. The curve passes through (11, 4·83), (11, −·83), (12, 6·57), (12, −2·57), (−1, −4·83), (−1, ·83), (−2, −6·57), (−2, 2·57).

**9.** An ellipse with (2, −3) as centre and $x-y=5$, $x+y=-1$ as axes. Semi-axes $=\sqrt{2}$ or 1·41, $\frac{1}{3}\sqrt{6}$ or ·82. Intercepts on $OX$ and $OY$ are imaginary. The curve passes through (2, −4), (2, −2), (1, −3), (3, −3).

**10.** An hyperbola with $2x-y+1=0$ and $x+2y-2=0$ as axes. Semi-axes = 3 and $\frac{3}{2}$. Centre is (0, 1). Asymptotes are $x=0$, $3x-4y+4=0$. Intercepts on $OX$ are imaginary, on $OY=\infty$. Curve passes through (1, 4), (2, 3·62), (3, 4), (·5, 5·87), (−1, −2), (−2, −1·62), (−3, −2), (−·5, −3·87).

**11.** A hyperbola with $4x-y+12=0$, $x-3y+9=0$ as asymptotes. Centre is $(-\frac{27}{11}, \frac{24}{11})$. One branch of the curve lies on same side of both asymptotes as the origin. Intercepts on $OX=-10\cdot36$, $-1\cdot64$; on $OY=13\cdot3$, $1\cdot7$. The curve passes through $(-2, 6\cdot91)$, $(-2, -\cdot58)$, $(-3, 4\cdot8)$, $(-3, -2\cdot8)$, $(-4, 3\cdot45)$, $(-4, -4\cdot95)$, $(-5, -9\cdot27)$, $(-5, 2\cdot6)$, $(1, 17)$, $(1, 2\cdot33)$.

**12.** Two straight lines $4x+5y-7=0$, $2x-3y+5=0$, intersecting at $(-\frac{2}{11}, \frac{17}{11})$. Intercepts on $OX=-\frac{5}{2}, \frac{7}{4}$; on $OY=\frac{5}{3}, \frac{7}{5}$.

**13.** A parabola, axis $x+1=0$, tangent at vertex $3y+2=0$, latus rectum $=3$. Curve and origin are on opposite sides of tangent at vertex. Intercepts on $OX$ imaginary; on $OY=-1$. Curve passes through $(-3, -2)$, $(1, -2)$.

**14.** A parabola, axis $2x-3y+\frac{17}{11}=0$, tangent at vertex, $3x+2y-\frac{28}{33}=0$, latus rectum $=\cdot45$. Curve and origin are on same side of tangent at vertex. Intercepts on $OX=-1\cdot87$, $-\cdot13$; on $OY=\cdot33$. Curve passes through $(-1, -1)$, $(-1, \cdot33)$, $(-2, -1\cdot94)$, $(-2, -\cdot06)$.

**15.** A hyperbola with $(-1, 0)$ as centre and $1\cdot62x-y+1\cdot62=0$, $x+1\cdot62y+1=0$ as axes. Semi-transverse axis $=\cdot79$, semi-conjugate axis $=1\cdot27$. Asymptotes are $y=0$, $y+2x+2=0$. Intercepts on $OX=\infty$; on $OY=-2\cdot41$, $\cdot41$. The curve passes through $(1, -4\cdot24)$, $(1, \cdot24)$, $(2, -6\cdot16)$, $(2, -\cdot16)$, $(-1, \pm1)$, $(-2, 2\cdot41)$, $(-2, -\cdot41)$.

**16.** A hyperbola with $(-\cdot2, 1\cdot4)$ as centre and $2x+y-1=0$, $x-2y+3=0$ as axes. Length of semi-transverse axis $=2$, length of semi-conjugate axis $=1\frac{1}{3}$. Asymptotes are $8x-y+3=0$, $4x+7y-9=0$. Intercepts on $OX$ imaginary; on $OY=5\cdot61$, $-1\cdot34$. The curve passes through the points $(1, 12\cdot01)$, $(1, -\cdot3)$, $(2, 19\cdot58)$, $(2, -\cdot44)$, $(-1, -6\cdot39)$, $(-1, 3\cdot24)$, $(-2, 3\cdot42)$, $(-2, -12\cdot56)$.

**17.** A parabola with $y+2=0$ as axis and $x-\frac{1}{6}=0$ as tangent at vertex. Latus rectum $=6$. The curve is on the origin side of the tangent at the vertex. Intercepts on $OX=-\frac{1}{2}$; on $OY=-1, -3$.

**18.** A hyperbola with $4x-y+12=0$, $x-3y+9=0$ as asymptotes. Centre is $(-\frac{27}{11}, \frac{24}{11})$. The origin and curve are in same angle of the asymptotes. Intercepts on $OX=0, -12$; on $OY=0$, $15$. Curve passes through $(2, 21\cdot97)$, $(2, 1\cdot7)$, $(5, 33\cdot25)$, $(5, 3\cdot4)$, $(-2, 9\cdot2)$, $(-2, -2\cdot9)$, $(-5, -10\cdot9)$, $(-5, 4\cdot3)$.

**19.** Two straight lines $x-y+1=0$, $x+y-3=0$ intersecting at $(1, 2)$.

**20.** A parabola with $2x-3y+4=0$ as axis and $3x+2y+16=0$ as tangent at vertex. Latus rectum $=\dfrac{1}{\sqrt{13}}=\cdot 28$. The curve is on the origin side of the tangent at the vertex. Intercepts on $OX = 0, -3\cdot 25$; on $OY = 0, 2\cdot 89$. Curve passes through $(-2, 1\cdot 17)$, $(-2, -\cdot 95)$.

**21.** A circle with $(5, -3)$ as centre and radius $= 6$. Intercepts on $OX = 10\cdot 20, -\cdot 20$; on $OY = -6\cdot 32, \cdot 32$.

**22.** A rectangular hyperbola with $(\cdot 6, \cdot 8)$ as centre, and $3x-y-1=0$, $x+3y=3$ as axes. Semi-axes $= 1\cdot 26$. Asymptotes are $2x+y-2=0$, $x-2y+1=0$. Intercepts on $OX$ imaginary; on $OY = 2\cdot 85, -\cdot 35$. Curve passes through $(1, 2)$, $(1, -1)$, $(2, 2)$, $(2, -2\cdot 5)$, $(-1, 4\cdot 45)$, $(-1, -\cdot 45)$.

**23.** Two parallel straight lines $2x-3y-3\cdot 45=0$, $2x-3y+1\cdot 45=0$. Intercepts on $OX = 1\cdot 72, -\cdot 72$; on $OY = -1\cdot 15, \cdot 48$.

**24.** A parabola with $3x+4y-3=0$ as axis and $8x-6y-47=0$ as tangent at vertex. Latus rectum $= \cdot 4$. The curve is on the origin side of the tangent at the vertex. Intercepts on $OX = 2\cdot 69, -1\cdot 58$; on $OY = 2\cdot 74, -\cdot 87$. The curve passes through $(1, 1\cdot 76)$, $(1, -1\cdot 38)$, $(2, \cdot 74)$, $(2, -1\cdot 86)$, $(3, -\cdot 36)$, $(3, -2\cdot 27)$.

**25.** Two straight lines $5x+4y-3=0$, $3x-7y+2=0$ intersecting at $(\frac{13}{47}, \frac{19}{47})$. Intercepts on $OX = \frac{3}{5}, -\frac{2}{3}$; on $OY = \frac{3}{4}, \frac{2}{7}$.

**26.** Two imaginary parallel straight lines $2x+6y+5=\pm\sqrt{-23}$.

**27.** An ellipse with $(\frac{1}{13}, \frac{5}{13})$ as centre, $2x-3y+1=0$, $3x+2y-1=0$ as axes. Lengths of semi-axes $= 2, 1$. Intercepts on $OX = -1\cdot 58, 1\cdot 18$; on $OY = 1\cdot 49, -\cdot 79$.

**28.** A rectangular hyperbola with $x+y-2=0$, $x-y+4=0$ as asymptotes. Centre is $(-1, 3)$. Axes are $x+1=0$, $y-3=0$. Length of semi-axes $=\sqrt{20}=4\cdot 47$. Intercepts on $OX$ imaginary; on $OY = 7\cdot 57, -1\cdot 57$. Curve passes through $(-2, 7\cdot 57)$, $(-2, -1\cdot 57)$, $(1 \text{ or } -3, 7\cdot 90)$, $(1 \text{ or } -3, -1\cdot 90)$, $(3 \text{ or } -5, 9)$, $(3 \text{ or } -5, -3)$.

**29.** A parabola with $7x+9y+1=0$ as axis, and $9x-7y+11=0$ as tangent at vertex. Latus rectum $=\dfrac{1}{\sqrt{130}}=\cdot 088$. Curve and origin are on opposite sides of tangent at vertex. Intercepts on $OX$ and $OY$ imaginary. Curve passes through $(-1, \cdot 9)$, $(-1, \cdot 52)$, $(-2, 1\cdot 95)$, $(-2, 1\cdot 03)$.

**30.** Two parallel straight lines $x+2y+1=0$, $x+2y+2=0$. Intercepts on $OX = -1, -2$; on $OY = -\frac{1}{2}, -1$.

**31.** A rectangular hyperbola with origin as centre and $15x+8y=0$, $8x-15y=0$ as axes. Lengths of semi-axes $=1$. Asymptotes are $y=\frac{23}{7}x$, $y=-\frac{7}{23}x$. Intercepts on $OX$ imaginary; on $OY=\pm 1{\cdot}34$. Curve passes through $(1, 3{\cdot}73)$, $(-1, -3{\cdot}73)$, $(1, -{\cdot}75)$, $(-1, {\cdot}75)$.

**32** A hyperbola with $(-1, -2)$ as centre and $3x+2y+7=0$, $2x-3y-4=0$ as axes. Length of semi-transverse axis $=1$, of semi-conjugate axis 2. Asymptotes are $7x-4y-1=0$, $x-8y-15=0$. Intercepts on $OX=15{\cdot}49$, $-{\cdot}34$; on $OY={\cdot}45$, $-2{\cdot}57$. Curve passes through $(1, -2{\cdot}19)$, $(1, 1{\cdot}94)$, $(-1, -3{\cdot}27)$, $(-1, -{\cdot}72)$, $(-2, -4{\cdot}45)$, $(-2, -1{\cdot}42)$, $(-3, -5{\cdot}94)$, $({\cdot}3, -1{\cdot}81)$.

**33.** An ellipse with $(2\sqrt{3}-\frac{3}{2}, -2-\frac{3}{2}\sqrt{3})$, *i.e.* $(1{\cdot}96, -4{\cdot}60)$ as centre, and $x+\sqrt{3}\,y+6=0$, $\sqrt{3}\,x-y-8=0$ as axes. Semi-axes $=5$ and 3. Intercepts on $OX$ imaginary; on $OY=-6{\cdot}91$, $-{\cdot}99$. The curve passes through $(-2, -4{\cdot}93)$, $(-2, -1{\cdot}65)$, $(-1, -6{\cdot}11)$, $(-1, -1{\cdot}12)$, $(2, -7{\cdot}9)$, $(2, -1{\cdot}37)$, $(4, -8{\cdot}21)$, $(4, -2{\cdot}34)$, $(6, -7{\cdot}5)$, $(6, -4{\cdot}37)$.

**34.** An ellipse with $(1, 1)$ as centre, and $x-\sqrt{3}\,y+(\sqrt{3}-1)=0$, $\sqrt{3}\,x+y-(\sqrt{3}+1)$ as axes. Lengths of semi-axes $=\sqrt{3}$, $\sqrt{2}$, Intercepts on $OX=2{\cdot}04$, $-{\cdot}41$; on $OY=2{\cdot}02$, $-{\cdot}31$.

**35.** An ellipse with $(8, 4)$ as centre, and $2{\cdot}414x-y-15{\cdot}312=0$, ${\cdot}414x+y-7{\cdot}312=0$ as axes. Semi-axes $=8{\cdot}6$, $2{\cdot}46$. Axis of $x$ is a tangent at $x=14$, and axis of $y$ is a tangent at $y=7$.

**36.** An ellipse with $(-1, 8)$ as centre, and $(1+\sqrt{2})\,x+y-(7-\sqrt{2})=0$, $x-(\sqrt{2}+1)\,y+(9+8\sqrt{2})$ as axes. Semi-axes $=7{\cdot}39$, $3{\cdot}06$. Intercepts on $OX$ imaginary; on $OY=1{\cdot}52$, $12{\cdot}48$. Curve passes through $(-2, 3{\cdot}71)$, $(-2, 14{\cdot}29)$.

**37.** An ellipse with $3x-y+3=0$, $x+3y=0$ as axes. Centre is $(-{\cdot}9, {\cdot}3)$. Lengths of semi-axes $=2$, 1. Intercepts on $OX=-2$, ${\cdot}054$; on $OY=2$, $-{\cdot}154$.

## EXAMINATION PAPER III. (PAGE 139.)

**1.** §§ 94–96.

**2.** $85x^2+30xy+45y^2-38x-66y-335=0$

and $45x^2-30xy+85y^2+12x-116y-320=0$.

**10.** $h^2=ab$ and $af^2+bg^2=2fgh$.

## CHAPTER XII. (PAGES 140–168.)

**3.** $3x+y-4=0$.

**4.** $x+y-3=0$, $2x+y+4=0$.

**5.** $x=\pm\sqrt{3}$, $y=\mp\sqrt{3}$.

**9.** $(-\frac{17}{5}, -\frac{5}{4})$.

**13.** $45°$ and $\tan^{-1}\frac{1}{3}$ with axis; $(-2, 0)$, $(-10, -2)$.

**14.** Line is $x+y=5$; curve is a parabola; $(-\frac{31}{2}, \frac{41}{2})$.

**15.** $x^2/a^2-y^2/b^2=0$;

$$-3(4x-y+\tfrac{3}{2})^2+2(4x-y+\tfrac{3}{2})(-x-3y+1)+4(-x-3y+1)^2=0$$
or $\quad 208x^2-104xy-156y^2+156x+104y-1=0.$

**18.** (i.) A straight line; (ii.) a conic. **19.** Internally; 13 : 3.

**20.** $2x+y=7$. **21.** 3 : 2, 3 : 4, both externally; (2, 2), $(3, \frac{3}{2})$.

**24.** $x-2y-1=0$, $6x-8y-9=0$, $x-y-2=0$.

**25.** $\pm 2\sqrt{2}x+3y+6=0$, $\pm\sqrt{5}x+3y+3=0$, $\pm\sqrt{13}\,x+3y+9=0$.

**26.** $\pm\sqrt{3}x\pm 2y=4$, $x\pm y+2p=0$, $\pm 3x\pm\sqrt{10}\,y=9$.

**27.** $x+y+2=0$, $x+y-\sqrt{26}=0$.

**29.** $x(x'+y'\cos\omega+g)+y(x'\cos\omega+y'+f)+gx'+fy'+c=0$.

**30.** $299x^2+2310xy+2247y^2+2772x-2940y-22596=0$.

**31.** $x(x-y+p)=0$. **32.** $b^2x^2-8a^2y^2=0$.

**33.** (*a*) $28x^2+384xy+131y^2+512x+864y+96=0$;
(*b*) $3x^2+2xy+8y^2-8x-18y+13=0$; two imaginary straight lines.

**34.** $121x^2-156xy+24y^2+70x+60y-95=0$; $\tan^{-1}(\cdot 78)$.

**36.** $xx_1/a^2+yy_1/b^2=1$. **37.** $yy_1=2a(x+x_1)$.

**38.** $y_1x+x_1y=2c^2$ or $x/x_1+y/y_1=2$.

**39.** $x(ax_1+hy_1)+y(hx_1+by_1)=1$.

**40.** $x(ax_1+hy_1+g)+y(hx_1+by_1)+gx_1=0$. **41.** $\frac{1}{3}\sqrt{30}$.

**43.** $n^2a^2+m^2b^2=m^2n^2$. **44.** $\sqrt{3}x-y=\pm\sqrt{13}$.

**45.** $\frac{1}{2}\sqrt{a^2+3b^2}$. **49.** $A^2m^2+2Alp+Bp^2=0$.

**50.** (i.) $-2\cdot 6$, $-\cdot 21$; (ii.) $-3\cdot 52$, $\cdot 6$; (iii.) $4\cdot 2$, $-1\cdot 98$.

**53.** $x\cos\theta+y\sin\theta=a(1+\cos\theta)$. **54.** $y'y''/p$, $\frac{1}{2}(y'+y'')$.

**56.** (1, 2), $\frac{1}{2}\sqrt{2}$. **57.** $y^2-4ax=a^2n^2$. **58.** $\tan^{-1}\frac{1}{2}$, $2\sqrt{5}$.

**61.** When $a<b$ two perpendicular tangents cannot be drawn.

**62.** $y^2=(p^2/q^2+q^2/p^2+2)\,ax$. **65.** $2x_1x_2/(x_1+x_2)$; $2y_1y_2/(y_1+y_2)$.

**68.** $2x-3y+9=0$, $(-\frac{15}{2}, -2)$.

**75.** $x^2\tan^2\alpha-y^2+2ax(\tan^2\alpha+2)+a^2\tan^2\alpha=0$.

## Chapter XIII. (Pages 169–191.)

**1.** (i.) $3x+y+2=0$; (ii.) $2x+y+1=0$.

**9.** $2(1+mm')+m+m'=0$. **12.** $a^2A+b^2B=0$. **16.** $y=\pm bx/a$.

**19.** $CP=\sqrt{5}$; $p=\frac{7}{10}\sqrt{10}$; $\sin\omega=\frac{7}{10}\sqrt{2}$. **20.** $\pm\sqrt{3}$, $\mp\sqrt{3}$.

**22.** $2\cdot 18$, $\cdot 45$. **23.** $3\cdot 92$, $1\cdot 62$.

**25.** $\frac{1}{2}x^2-y^2+1=0$; $3x^2-y^2+2=0$; $Ax^2+2Hxy+By^2+C=0$;
$xy+2x+3y+11=0$.

**26.** $x^2+2xy-y^2+2x+4y+\frac{1}{2}=0$; $x^2+2xy-y^2+2x+4y+1=0$.
**28.** $3x+4y=0$. **31.** 1·52, ·56.
**32.** $CP=\sqrt{2}$, $CD=\frac{1}{6}\sqrt{78}$, $\sin^{-1}(\frac{1}{26}\sqrt{26})$.
**33.** $y=\frac{3}{2}$; $(\frac{3}{4},\frac{3}{2})$. **34.** $y=1$; $(-\frac{1}{2},1)$. **36.** $3x+3y+2=0$.
**38.** $xy+2gx+2fy+8fg-c=0$.
**40.** $a^2\sin\theta\,(x_1\sin\theta-y_1\cos\theta)/(a^2\cos^2\theta+b^2\sin^2\theta)$,
$-b^2\cos\theta\,(x_1\sin\theta-y_1\cos\theta)/(a^2\cos^2\theta+b^2\sin^2\theta)$.
**41.** $9AA'+4BB'=0$. **44.** $a^2x^2+b^2y^2=a^2(a^2-b^2)$.
**45.** $x+y=a+b$. **46.** $y^2=2a(x-a)$. **47.** $a+h(m+m')+bmm'=0$.
**48.** $Ax+By=0$, $Bx+Cy=0$. Condition is $B^2=AC$. All straight lines joining two given parallel straight lines are bisected by the line parallel to the given lines and midway between them.
**49.** $\sqrt{2}a\sin\frac{1}{2}\omega$, $\sqrt{2}a\cos\frac{1}{2}\omega$. **52.** $x^2/a^2+y^2/b^2=\frac{1}{2}$. **53.** Yes.
**57.** $ax^2+2hxy+by^2+gx+fy=0$.
**63.** $3(x/a^2+my/b^2)^2+(1/a^2+m^2/b^2)(x^2/a^2+y^2/b^2-1)=0$.

## Chapter XIV. (Pages 192–202.)

**1.** $5x-3y=2$. **2.** $6x-3y=2$.
**4.** $h(x'^2-y'^2)=(a-b)x'y'$. Origin is centre of curve, and normal passes through centre only when $(x',y')$ is on an axis.
**11.** $4\sqrt{a(a+x')^3}/x'$. **13.** $\frac{1}{\sqrt{10}}$, $\frac{3}{\sqrt{10}}$. **14.** $\frac{1}{\sqrt{5}}$, $\frac{2}{\sqrt{5}}$.
**16.** $(1,2)$, $(\frac{1}{4},1)$, $(\frac{9}{4},-3)$. **17.** $a^2/\sqrt{a^2+b^2}$, $b^2/\sqrt{a^2+b^2}$.
**20.** (i.) $3x-3\sqrt{3}y=7a$; (ii.) $x-y=3a$; (iii.) $\sqrt{3}x+y=5\sqrt{3}a$
**21.** $x-y=4a$. **23.** $p^2(a^2\sin^2\alpha+b^2\cos^2\alpha)=(a^2-b^2)^2\cos^2\alpha\sin^2\alpha$.
**24.** $a^2y/(a^2-b^2)$, $\pm b\sqrt{(a^2-b^2)^2-a^2y^2}/(a^2-b^2)$.
**25.** $x'(1-m^2/a^2)$, $y'(1-m^2/b^2)$. **26.** $4\sqrt{2}a$. **27.** $y^2=4a(x-4a)$.
**29.** (i.) A finite quantity = zero. No such line can be normal to the circle $x^2+y^2=a^2$. **30.** $a^2/l^2-b^2/m^2=(a^2+b^2)^2$.
**31.** $y^2=a(x-a)$. **33.** $y^2=a(x-a)$.

## Examination Paper IV. (Page 203.)

**1.** $x(\frac{1}{2}mq-g)+y(\frac{1}{2}mp+bq+f)-gp+fq-c=0$.
**3.** $x^2-y^2+6ax+a^2=0$. **4.** $10x+2y+25=0$.
**5.** $3x^2-4xy+2x-3y-7\frac{3}{8}=0$.

## Chapter XV. (Pages 204–219.)

**1.** $x+y-2=0$. **2.** $14x+13y+17=0$. **3.** $gx+fy+c=0$.
**5.** The normal. **11.** $\frac{9}{10}$, $-\frac{12}{5}$. **12.** 1, $\frac{1}{2}$.
**13.** $-\frac{1}{2}$, $\frac{1}{2}$. **17.** $Bl^2-2Hlm+Am^2=AB-H^2$. **18.** $x-y=1$.
**19.** $3x+3y+5=0$, $4x+5y-2=0$, $x-2y=a^2$,
$x(a+2h+g)+y(h+2b)+g=c$
**20.** $(a^2l,\ -b^2m)$, $(-1/l,\ -2am/l)$, $(2c^2m,\ 2c^2l)$.
**21.** $a^2ll'-b^2mm'=1$, $l+l'+2amm'=0$, $2c^2(lm'+l'm)=1$.
**30.** $-p\sec\alpha$, $-2a\tan\alpha$.
**31.** Diameter conjugate to the given direction. **33.** $y^2=2ax-a^2$.
**34.** $x^2-y^2=4a^2$. **36.** $f/(f^2-g^2)$, $-g/(f^2-g^2)$; $x^2-y^2=lx-my$.
**39.** The radical axis. [Take this line for axis of $y$; then the circles are $x^2+y^2+2gx+c=0$ and $x^2+y^2+2g'x+c=0$.]

## Chapter XVI. (Pages 220–242.)

**1.** $(y-1)=(x-1)^2$. **2.** $(x-c)^2/a^2+(y-d)^2/b^2=1$.
**4.** (i.) 45°, &c.; (ii.) $\sin^{-1}(b/a)=\cos^{-1}e$, &c.
**7.** $(1,2)$, $(4,4)$, $(\frac{1}{4},-1)$, $(\frac{9}{4},-3)$. **8.** 1. **9.** $\frac{1}{12}$, $\frac{1}{6}\sqrt{3}$.
**10.** $(\mu_1+\mu_2)y-2x=2a\mu_1\mu_2$. **13.** $AG=2a+a\mu^2$.
**14.** $a(\mu^2+2)^2/\mu^2$, $-2a(\mu^2+2)/\mu$. **15.** $\mu=1, 2, -3$.
**23.** $x(\sqrt{3}-1)/a+y(\sqrt{3}+1)/b=2$, $x/a+y/b=\frac{1}{2}(\sqrt{3}+1)$.
**34.** Extremities of axes. **38.** $\left\{-a\pm\sqrt{a^2-mc}\right\}/am$; $c=a/m$.
**42.** $y^2-4ax=(a+x)^2\tan^2\alpha$. **49.** $2(x+y-1)=(x-y)^2$.
**52.** $x^2/a^2+y^2/(2b^2)=1$. **59.** $x+a\cot(\alpha-\beta)=0$, $y-b\cot(\alpha-\beta)=0$.
**63.** $(a'x-ay)^2(b'x-by)+(a'b-ab')^3=0$.

## Chapter XVII. (Pages 243–253.)

**5.** $3\frac{3}{4}$ or $7\frac{1}{2}$. **11.** A circle. **14.** $a=l/(e^2-1)$, $b=l/\sqrt{e^2-1}$.
**17.** The latus rectum and any two focal chords equally inclined to it.
**20.** $(lA-e)^2+l^2B^2=1$. **23.** $r\cos\theta=3a$, a straight line perp. to axis.

## Examination Paper V. (Page 254.)

**3.** $(\frac{4}{3}, \frac{5}{6})$.

## Chapter XVIII. (Pages 255–272.)

**4.** $21x^2+3xy+7y^2=43$. **5.** $c'S-cS'=0$.
**6.** $3x^2+2xy+3x=3$. **8.** $17x^2+26xy+17y^2-60x-60y+60=0$.

**14.** $3x^2-xy-5y^2+x+y+1=0$. **15.** $5x^2-6xy-6y^2+9x-3y=0$.
**20.** $\sqrt{x}+\sqrt{y}=\sqrt[4]{8a^2}$ or $x^2+y^2-2xy-4\sqrt{2a}(x+y)+8a^2=0$.
**24.** $2x^2-3xy+2y^2-2x-4y=0$. **25.** $x(x-y-1)=0$.
**26.** $(\alpha-\beta)(\gamma-\delta)/\alpha\beta\gamma\delta$.
**31.** $x^2/\cos^2\alpha-y^2/\sin^2\alpha=a^2-b^2$. **32.** $5(x-y)=1$.
**45.** $A/(x-2)+B/(y+1)+C/(x+y)=0$. **46.** $x^2+y^2-3x+3y+4=0$.
**47.** Circle $x^2+y^2=a^2+b^2$, where $a$ and $b$ are semi-axes of ellipse.
**48.** $m/(2-kmn)$, $n/(2-kmn)$. **51.** $x=\pm\frac{1}{2}\sqrt{2a}$, $y=\frac{1}{2}\pm\frac{1}{2}\sqrt{2b}$.
**56.** Use § 237. **57** $x^2+pxy-y^2=a^2-b^2$, where $p$ is constant.
**59.** $\pm\alpha/a\pm\beta/b=1$.
**62.** (Use $p^2=a^2\cos^2\alpha+b^2\sin^2\alpha$.) $x^2/(R^2-b^2)+y^2/(R-a^2)=1$, where $R$ is radius of circle. **63.** $x/(a\cos^2\theta)+y/(b\sin^2\theta)=1$.

## Chapter XIX. (Pages 273–282.)

**2.** $1/(2\sqrt{AB})$ along the coordinate axes. **4.** $y^2+16ax=0$.
**7.** $y^2+4ax=4a$. **8.** $(\lambda x-\mu y)^2=\mu c(y-c)$. **9.** $4y^3=9cx^2$.
**12.** $(x+y-c)^2+cy=0$. **13.** $y(y-2b)=0$, where $b$ is radius of circle.
**14.** $x^2/a^2+y^2/b^2=\frac{1}{8}$. **15.** $(x-b)^2+(y-c)^2=a^2$.
**22.** $x^2/a^2(1+b^2/a^2)+y^2/b^2(1+a^2/b^2)=1$.

## Chapter XX. (Pages 283–299.)

**2.** The first and second are harmonic with the third and fourth.
**4.** $\frac{5}{2}$, 2, $\infty$, 4, $\frac{7}{2}$, $\frac{10}{3}$, $\frac{11}{4}$, $\frac{14}{5}$, $\frac{17}{6}$, $\frac{20}{7}$, $\frac{23}{8}$, $\frac{26}{9}$. **8.** $\frac{1}{3}$. **9.** 5.
**10.** $x^2-7x+11=0$ or $x=\frac{1}{2}(-7\pm\sqrt{5})$. **19.** 4.
**23.** $3k^2-2l^2=0$. **29.** $a=b$. **38.** A particular case of Ex. 36.
**39.** If $P$, $Q$ are ends of diameter, $AP$, $AQ$ bisect angles at $A$.
**40.** Change into polar coordinates and find $OR$ in terms of $OP$. $OQ$, $OS$.
**41.** Change into polar coordinates and use $2/OP=1/OP_1+1/OP_2$.

## Examination Paper VI. (Page 300.)

**2.** The points of intersection of $ax^2+by^2+2hxy+2fy+c=0$ and $gx=fy$. **3.** $x^2+y^2-3x-3y+4=0$.
**6.** $xy=\pm c^2$. **7.** $4(x-2a)^3=27y^3$.

## Chapter XXI. (Page 301–307.)

**5.** $-\frac{1}{2}$
**6.** It is the circle on the line $FF'$ as diameter, $FF'$ being defined as in § 287. The point $O$ lies on the radical axis of any two circles through $AA'$ and $BB'$.

## Problem Paper 1. (Page 308.)

**1.** $\{(l^2-m^2)\,q-2lmp-2mn\}/(l^2+m^2)$,
$\{(m^2-l^2)\,p-2lmq-2ln\}/(l^2+m^2)$.

**2.** If base be taken as axis of $x$ and the mid-point of base as origin, $e=\frac{1}{3}\sqrt{6}$ and foci are $(0,\ \sqrt{3}a\pm\frac{2}{3}\sqrt{6}c)$.

**7.** Taking given line as axis of $x$ and perpendicular to it through centre of given circle as axis of $y$, equation of given circle is of the form $x^2+(y-b)^2=c^2$. Required locus are the two parabolas $x^2-2\,(b\pm c)\,y+b^2-c^2=0$.

**10.** $(\frac{1}{2}a,\ \frac{1}{2}a)$; $x+y=0$.

## Problem Paper 2. (Page 309.)

**1.** A circle $x^2+y^2-2x\,(x_1+x_2-2x_3)-2y\,(y_1+y_2-2y_3)$
$+x_1^2+y_1^2+x_2^2+y_2^2-x_3^2-y_3^2=0$,
where the vertices are $(x_1,y_1)$, $(x_2,y_2)$, $(x_3,y_3)$. **5.** $(8,4)$; $8\sqrt{7}\,\pi$.

**8.** Take directrix as axis of $y$ and focus on axis of $x$, distance $a$ from directrix. Then locus is $y^2-mxy-ax+a^2=0$, where $m$ is the tangent of the angle the given line makes with axis of $x$.

## Problem Paper 3. (Page 310.)

**1.** $b^2=4ac,\ bd=2ae,\ d^2=4af.$ **2.** $2,\ (3,\ 60°)$. **3.** $\pm\pi F/(AC-1)$.
**4.** $ax\pm by=\sqrt{a^4+a^2b^2+b^4}$. **7.** $\frac{1}{2}ab\sin(\alpha-\beta)$.
**10.** $5x^2\pm6by-6b^2=0$; $y=\pm b\,(1-6b^2/5a^2)$.

## Problem Paper 4. (Page 311.)

**1.** $(5-4b/a)\,x^2-8xy+(5-4a/b)\,y^2=0$; $a=2b$ or $\frac{1}{2}b$.

**10.** Take one side as axis of $x$, vertex of triangle as origin; then locus is $x\,(m+n\cos\alpha)+y\,.\,n\sin\alpha=\frac{1}{2}c$.

## Problem Paper 5. (Page 312.)

**1.** $(4ac-b^2)/c$. **2.** $x^2+y^2-5dx+(5d\cos\alpha\pm4d)\,y/\sin\alpha+4d^2=0$.
**4.** $x^2+y^2=a^2b^2/(a^2+b^2)$. **6.** $bx\sin\frac{1}{2}\gamma=ay\cos\frac{1}{2}\gamma$, where $\gamma$ is const.
**8.** $7x^2+7y^2-39x-15y+32=0$.
**9.** Take given point as origin, and circle $x^2+y^2+2gx+2fy+c=0$; envelope is $x^2\,(2c-f^2)+2fgxy+y^2\,(2c-g^2)+2cgx+2cfy+c^2=0$.
**10.** $y^3-4axy+24a^3=0$.

## Problem Paper 6. (Page 313.)

**1.** With $AB$ as axis of $x$ and its middle point as origin, the locus of $C$ is $8x^2+4cx-y^2=0$.

**2.** $\frac{1}{2}(11\sqrt{3}-5)c$.

**3.** $a^{\frac{1}{3}}x+b^{\frac{1}{3}}y+a^{\frac{2}{3}}b^{\frac{2}{3}}=0$.

**10.** $x^2+y^2\pm 2cx/\sqrt{1-4k^2}+c^2=0$, where $k$ is the ratio and $2c$ the base, referring to base as axis and its mid-point as origin.

## Problem Paper 7. (Page 314.)

**1.** $bx-ay+ak-bh=0$;
$(b^2h-abk-ac)/(a^2+b^2)$, $(a^2k-abh-bc)/(a^2+b^2)$.

**3.** $x^2+y^2+(x+y)(C+D)/(A+B)+\frac{1}{2}(C+D)^2/(A+B)^2$
$=\frac{1}{4}(C-D)^2/(A^2+B^2)$.

**7.** $2x/a+1=(3x+a)^2/y^2$.

## Problem Paper 8. (Page 315.)

**6.** Centre is $(1,2)$; lengths of axes, $1$, $\sqrt{3}$; equations of axes referred to centre, $x-y=0$, $x+y=0$; asymptotes, referred to original axes, $x^2-4xy+y^2+6x-3=0$.

**9.** $(x^2+y^2)(bx+ay)+(ax-by)^2=0$, where $x^2=4ay$ and $y^2=4bx$ are the parabolas.

## Problem Paper 9. (Page 316.)

**2.** A circle.

**3.** $\dfrac{r_2r_3\sin(\theta_2-\theta_3)+r_3r_1\sin(\theta_3-\theta_1)+r_1r_2\sin(\theta_1-\theta_2)}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)}}$.

**4.** $(2,-3)$; $x^2-xy+y^2=1$.

**10.** $x^2-y^2-ax+by=0$.

## Problem Paper 10. (Page 317.)

**1.** $ax-by-a^2=0$, $ax-by+b^2=0$, $bx+ay=ab\pm(a^2+b^2)$; point of intersection is $\left\{\frac{1}{2}(a\pm b),\ \frac{1}{2}(b\pm a)\right\}$.

**2.** $2x\left\{a(r_2-r_3)+c(r_3-r_1)+e(r_1-r_2)\right\}+2y\left\{b(r_2-r_3)+d(r_3-r_1)+f(r_1-r_2)\right\}$
$=(a^2+b^2)(r_2-r_3)+(c^2+d^2)(r_3-r_1)+(e^2+f^2)(r_1-r_2)+(r_2-r_3)(r_3-r_1)(r_1-r_2)$,
where the centres are $(a,b)$, $(c,d)$, $(e,f)$.

**8.** $2(x^2+y^2)-(a+b)x+(c-ab/c)y=0$, where the three points are $(a,0)$, $(b,0)$, $(0,c)$.

### Problem Paper 11. (Page 318.)

**1.** Take $CA$ and $CB$ as axes, and $CA = a$, $CB = b$; then locus is $2x/a + 2y/b = 1$. **2.** $\frac{8}{5}\sqrt{5}$; $\infty$. **5.** $\frac{8}{5}$.

### Problem Paper 12. (Page 319.)

**2.** $(2ff' + 2gg' - c - c')/2\sqrt{\{f^2 + g^2 - c)(f'^2 + g'^2 - c')\}}$.

**7.** $7x + 9y + 1 = 0$. **8.** $x + y = 0$.

### Problem Paper 13. (Page 320.)

**3.** $(\pm\frac{3}{4}\sqrt{6}\,a,\ \mp\frac{1}{4}\sqrt{6}\,a)$, $(\pm\frac{1}{3}\sqrt{6}\,a,\ \pm\sqrt{6}\,a)$.

**6.** $e = (a-b)/\sqrt{(a^2+b^2)}$; axes are $(a+b)/\sqrt{(2ab)}$, $(a+b)/\sqrt{(a^2+b^2)}$; latus rectum $= 2\sqrt{(2ab)}\,(a+b)/(a^2+b^2)$; equation to tangent is $\pm(a-b)\,x/\sqrt{(a^2+b^2)} \pm y = (a+b)/\sqrt{(2ab)}$. **10.** $k = 2$, $r = 2$.

### Problem Paper 14. (Page 321.)

**1.** $x(a - b\sqrt{3}) - y(a\sqrt{3} - b) + ab = 0$. **8.** $\frac{3}{8}$, $\frac{3}{8}$.

### Problem Paper 15. (Page 322.)

**2.** At ends of minor axis. **3.** $2\sqrt{590}\,a$. **4.** $x + a = 0$.

# INDEX.

The page references in italics are to Part I.

PAGES
ABBREVIATED notation for conics ... 255
Abridged notation for straight lines *34*
Abscissa ... *2*
Adams' Property of conics ... 251
Angle between two straight lines *25, 36*
,, , Eccentric ... 228
Anharmonic ratio ... 301
Area of ellipse ... 229
,, , triangle ... *8, 11*
Asymptotes ... 73, 80, 81
,, , Equation of 81, 84, 102–4, 146
Auxiliary Circle ... 164
Axes ... *2*
,, , Change of ... *39, 43–48*
,, , Equation of ... 96
,, , Length of ... 96–99
,, of conic ... 96
,, ,, ellipse ... 53
Axis of a parabola ... 38, 117

BISECTORS of angles between two lines ... *32, 37*

CARTESIAN Coordinates ... *2, 10*
Centre of a conic ... 91
Centre of ellipse ... 53
,, , Polar of ... 215
Change of axes ... *39, 43–48*
Chord bisected at a given point ... 172
,, of circle ... *9, 15*
,, ,, conic, ... 157, 223, 230, 248
,, ,, ellipse ... 230
,, ,, parabola ... 223
,, , Rectangle of segments of ... 147
Circle, Auxiliary ... 164
,, , Condition for ... 3
,, , Director ... 161
,, , Equation of ... 1–5
,, , Particular case of ellipse ... 63
Complete Quadrilateral ... 293
Concurrency, Condition of ... *27*

PAGES
Condition for circle ... 3
,, ,, ellipse ... 65, 102
,, ,, hyperbola ... 84, 102
,, ,, parabola ... 42, 46
,, ,, rectangular hyperbola 104
,, ,, two straight lines ... *40*
,, of tangency ... 13, 45, 64, 159
Confocal conics ... 264–270
,, parabolas ... 270
Conic, Pair of tangents to ... 155, 261
,, , Polar equation of ... 56, 73, 243
,, , Tangent to ... 142, 154, 157, 261
,, , touching axes ... 260
Conics, Harmonic properties of ... 295
,, through five points ... 263, 305
,, ,, four points 256–258, 264
Conjugate diameters ... 173–184, 196, 232
,, hyperbola ... 179–184
,, lines ... 212
,, points ... 212
Coordinates ... *1, 2, 9*
Corresponding points ... 228
Cross ratios ... 301–6

DEGREE of equation unaltered.. *47*
Diameters, conjugate 173–184, 196, 232
,, , equi-conjugate ... 177
Director Circle ... 161
Directrix ... 35, 50, 69
Distance between two points ... *3, 10*
Distance from a point in which a line meets a conic ... 140

ECCENTRIC Angle ... 228
Eccentricity ... 50, 69
Ellipse and hyperbola, Comparison between ... 73
,, , Area of ... 229
,, , Asymptotes of ... 103
,, , Centre of ... 53
,, , Chord of ... 230

The page references in italics are to Part I.

PAGES

Ellipse, Circle a particular case of 63
,, , Conditions for.............65, 102
,, , Definition of .............. .. 50
,, , Equation of .................50, 51
,, , ,, referred to axes parallel to principal axes 54
,, , Focal distances of........... 61
,, , Latus rectum of.............. 58
,, , Mechanical description of 62
,, , Normal to ..................193, 234
,, , Popular equation of......56, 243
,, , Shape of..................52, 56, 244
,, , Tangent to .................... 64
,, , Tracing of .................106–111
Envelopes ........................... 273–281
Equation of curve ...................... *12*
,, , Degree of, unaltered...... *47*
,, , Focus-pole .................. 243
,, , General ....................*20, 40*
,, of a conic referred to a tangent and normal 198
,, ,, ,, referred to conjugate diameters 178
,, ,, ,, referred to its centre ............ 92
,, ,, asymptotes 81, 102–104, 146
,, ,, chord of conic............ 157
,, ,, ,, ,, parabola ...... 223
,, ,, circle ........................ 1–5
,, ,, conjugate hyperbola... 180
,, ,, ellipse ..............50, 51, 54
,, ,, ,, referred to two conjugate diameters 178
,, ,, hyperbola ..............70, 85
,, ,, ,, referred to two conjugate diameters 184
,, ,, parabola ......36, 37, 40, 186
,, , Polar ...............56, 73, 243
Equi-conjugate diameters ............ 177
,, ,, , Equation of curve referred to 178
Equilateral Hyperbola................... 82

FOCAL distances ..................61, 78
Foci of involution ................ 306
Focus ................................35, 50, 69
Focus-pole equation .................... 243

GENERAL equation ...............*20, 40*
,, ,, , Reduction of 94
Geometrical construction of Polar 213

HARMONIC conjugates ............ 283
,, pair of lines ......... 288
,, ,, points ...... 287
,, pencil .................. 288
,, property of conics 296
,, property of quadrilateral ........... 292

PAGES

Harmonic range .......................... 283
Hyperbola and ellipse, Comparison between ................. 73
,, , Condition for .........84, 102
,, , Conjugate ...........179–184
,, , Conjugate diameters of 179, 235
,, , Definition of............... 69
,, , Equation of ............69, 70
,, , ,, referred to axes parallel to principal axes ...... 75
,, , ,, referred to asymptotes as axes.. 85
,, , ,, referred to two conjugate diameters ............ 184
,, , Equilateral ............... 82
,, , Focal distances of ...... 78
,, , Mechanical description of ........................ 79
,, , Normal to ............193, 236
,, , Polar equation of ...... 73
,, , Rectangular... ...... .... 82
,, , Shape of ...............71, 244
,, , Tangent to ............... 80
,, , Tracing of ...........112–116

INCLINATION of normal to major axis .......................... 195
Infinity, Line at ......................... 215
Intersection of straight lines .....*27, 28*
Invariants ................................*48*, 98
Inversion .................................. 28
Involution ................................ 306

LATUS rectum of parabola 38, 117, 120, 125, 247
,, ,, ellipse ......58, 247
,, ,, hyperbola ...77, 247
Limiting points .......................... 25
Line at infinity .......................... 215
,, divided in a given ratio ......... *5*
,, joining two points ............... *15*
Lines through origin...................... *36*
Loci ........................................ *12*
Locus of foot of perpendicular from focus on tangents...164, 165
,, ,, intersection of perpendicular tangents .........160–162

MAJOR axis .......................... 53
Mechanical description of ellipse ........................... 62
Mechanical description of hyperbola 79
,, ,, ,, parabola 35
Middle points of parallel chords 21, 169, 170, 185
Minor axis ................................ 53

The page references in italics are to Part I.

PAGES

NORMAL, Inclination to major axis ... 195
,, to circle ... 10
Normals to a conic ... 192
,, ,, ellipse ... 193, 234
,, ,, hyperbola ... 193, 236
,, ,, parabola ... 193, 197, 198, 226
,, in general ... 192
Notation, Abbreviated ... *34*, 255
,, , for equation of second degree ... *40*

OBLIQUE axes ... *3*
Ordinate ... *2*

PARABOLA, Axis of ... 38, 117
,, , Chord of ... 223
,, , Condition for ... 42, 46
,, , Definition of ... 35
,, , Equation of ... 36, 37
,, , ,, referred to tangent and diameter ... 186
,, , Equation, referred to axes parallel to principal axes ... 40
,, , Intersection, with any curve, ... 222
,, , Latus rectum of 38, 117, 120, 125, 247
,, , Mechanical description of ... 35
,, , Normal to 193, 197, 198, 226
,, , Shape of ... 37, 244
,, , Tangent to ... 45, 224
,, touching axes ... 260
,, , Vertex of ... 38
Parabolas, Confocal ... 270
,, , Plotting ... 39, 124–126
Parallel chords, Middle points of 21, 169, 170, 185
Parallelism of straight lines, Condition of ... *25*
Parameter ... 28, 220, 221, 273
Pascal's Theorem ... 305
Pencil ... 288, 302, 304
Perpendicular ... *23, 24*
,, , tangents, Locus of intersection of ... 160–162
Perpendicularity of straight lines, Condition of *25*
Point at infinity ... 286
,, circle ... 3
Polar Coordinates ... *9, 10*
,, equation of conic ... 56, 73, 243
,, , Geometrical construction of 213
,, of a point ... 16–18, 205, 206

PAGES

Polar of the centre ... 215
Pole of a straight line ... 205
,, , Geometrical construction of ... 213
Poles and polars, Reciprocal properties of ... 210–212
Projection ... 289

QUADRILATERAL, Complete ... 293

RADICAL Axis ... 22–24
Radius Vector ... *9*
Range, Harmonic ... 283
Ratio in which a conic divides a line 150
,, , Quadratic ... 150
Reciprocal properties of poles and polars ... 210–212
Rectangle of segments of chord of a conic ... 147
Rectangular axes, Change of ... *43*
,, hyperbola ... 82
,, ,, , Condition for ... 104
Reduction of general equation ... 94
Rules for tracing conics ... 127–129

SECOND focus and directrix ... 58, 76
Segment of chord, Rectangle of 147
Straight line, Equation of *14–18, 28, 29*
Symmetrical curve ... 37

TANGENCY, Conditions of 13, 45, 64, 159
Tangent to circle ... 7–9, 26
,, ,, ,, , length of ... 20
,, ,, conic ... 142, 154, 157, 249
,, ,, ellipse ... 64, 142
,, ,, hyperbola ... 80, 142
,, ,, parabola ... 45, 142, 224
Tangents from any point to a conic 155, 261
,, at right angles ... 160–162
Tracing of conics ... 106, 112, 124, 127
,, ,, ,, , Rules for ... 127–129
Triangle, Area of ... *8, 11*
Two straight lines, Angle between *25*
,, ,, , Bisectors of angles between *32, 37*
,, ,, , Condition for ... *40*

ULTIMATE intersections ... 274

VERTEX of parabola ... 38
Vectorial angle ... *9*

PRINTED AT THE BURLINGTON PRESS, FOXTON, NEAR CAMBRIDGE, ENGLAND.